# Cellular Homology and Mapping Tori

Algebraic Topology I | Fall 2026 | Companion to Lecture 8

Use the cellular homology theorem, the product-cell boundary formula, and the Wang sequence from Lecture 8. Give complete proofs of the deductions below; these three results need not be reproved. Coefficients lie in an abelian group $A$, fixed throughout, unless specified otherwise. There are four exercises.

For a finite cell complex $X$, write $\mathrm{sk}_n X$ for the union of its cells of dimension at most $n$, and put $\mathrm{sk}_{-1} X = \emptyset$. Its cellular chain group is

$$C_n^{\mathrm{cell}}(X;A) = \mathrm{H}_n(\mathrm{sk}_n X, \mathrm{sk}_{n-1} X; A).$$

An orientation of each cell identifies this group with one copy of $A$ per $n$-cell. The cellular differential is the connecting homomorphism followed by the map to relative homology of the preceding skeleton. For an integer $m$, write $A[m] = \ker(m : A \to A)$ and $mA = \mathrm{im}(m : A \to A)$.

## Exercise 1. Attaching One Cell by a Degree Map

Let $r \geq 1$ and $m \in \mathbb{Z}$. Let $f_m : S^r \to S^r$ be a map of degree $m$, meaning that its induced map on $\mathrm{H}_r(S^r;\mathbb{Z})$ is multiplication by $m$ after orientations are chosen. Form

$$M_{m,r} = S^r \cup_{S^r} D^{r+1}, \qquad \begin{array}{ccc} S^r & \xrightarrow{f_m} & S^r \\ \downarrow & & \downarrow \\ D^{r+1} & \longrightarrow & M_{m,r}. \end{array}$$

Use the cell structure on $S^r$ with one vertex and one $r$-cell. The attaching construction adds one $(r+1)$-cell.

(a) Write the cellular chain complex of $M_{m,r}$ with coefficients in $A$. Explain why its only potentially nonzero differential is multiplication by $m$ from degree $r+1$ to degree $r$. Treat $r = 1$ explicitly, so the role of the vertex is clear.

(b) Compute all the homology groups of $M_{m,r}$ in terms of $A$, $A/mA$, and $A[m]$. State the reduced groups when $m = 0$ and when $m = 1$.

(c) Take $m = 6$. Compute the homology first with integral coefficients, then with $\mathbb{Z}/2\mathbb{Z}$, and then with $\mathbb{Z}/5\mathbb{Z}$. Explain why tensoring the integral homology groups with $\mathbb{Z}/2\mathbb{Z}$ does not recover all the mod-two homology groups.

## Exercise 2. Projective Spaces and Change of Coefficients

Real projective space $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Its standard cell structure has one cell in every degree from $0$ to $n$. You may use the attaching-degree computation in Lecture 8: with suitable orientations, the cellular differential is multiplication by $2$ in positive even degrees and zero in odd degrees.

(a) Write the cellular complexes of $\mathbb{RP}^2$ and $\mathbb{RP}^3$ with coefficients in $A$, and compute their homology in every degree. Identify the homomorphism in each degree induced by the standard inclusion $\mathbb{RP}^2 \hookrightarrow \mathbb{RP}^3$.

(b) Specialize to $A = \mathbb{Z}$, $A = \mathbb{Z}/2\mathbb{Z}$, and $A = \mathbb{Z}/4\mathbb{Z}$. Identify explicitly the subgroup $A[2]$ in each case. Explain why the nonzero mod-two class in degree $2$ of $\mathbb{RP}^2$ is compatible with vanishing integral homology in degree $2$.

(c) Compute the relative groups $\mathrm{H}_k(\mathbb{RP}^3, \mathbb{RP}^2; A)$. Write the portion of the relative long exact sequence from degree $3$ through degree $1$, including the maps obtained in part (a). Check exactness for arbitrary $A$.

## Exercise 3. Product Cells and Their Signs

For oriented cells $e$ and $e'$ of dimensions $p$ and $q$, orient their product by placing the directions of $e$ before those of $e'$. The product-cell boundary formula is

$$\partial(e \otimes e') = \partial e \otimes e' + (-1)^p e \otimes \partial e'.$$

Give $S^1$ one vertex $v$ and one oriented edge $u$, so $\partial u = 0$. Give $\mathbb{RP}^2$ cells $e_0, e_1, e_2$ with $\partial e_2 = 2e_1$ and $\partial e_1 = 0$.

(a) List the product-cell bases for $\mathrm{C}_k^{\mathrm{cell}}(\mathbb{RP}^2 \times S^1; \mathbb{Z})$ in degrees $0, 1, 2, 3$. Write each differential in those bases, and verify directly that two consecutive differentials have zero composite.

(b) Compute $\mathrm{H}_k(\mathbb{RP}^2 \times S^1; A)$ in every degree. Compare the result with the circle-product splitting from Homework 7,

$$\mathrm{H}_k(Y \times S^1; A) \cong \mathrm{H}_k(Y; A) \oplus \mathrm{H}_{k-1}(Y; A),$$

where $\mathrm{H}_{-1}(Y; A) = 0$.

(c) In the reversed product $S^1 \times \mathbb{RP}^2$, compute $\partial(u \otimes e_2)$. Show on the product-cell bases that the interchange map is a chain map when its effect on a cell of bidegree $(p, q)$ includes the sign $(-1)^{pq}$. Explain the sign by interchanging the two blocks of oriented directions.

## Exercise 4. Mapping Tori of Circle Maps

For a continuous map $f : S^1 \to S^1$ of degree $d$, let $I = [0,1]$ and let $i_0, i_1 : S^1 \to S^1 \times I$ include the endpoints. Define its mapping torus by the coequalizer in topological spaces

$$S^1 \underset{i_0 \circ f}{\overset{i_1}{\rightrightarrows}} S^1 \times I \xrightarrow{\ q\ } T_f.$$

Thus $q \circ i_1 = q \circ i_0 \circ f$, and every continuous map $h : S^1 \times I \to Z$ satisfying $h \circ i_1 = h \circ i_0 \circ f$ factors uniquely through $q$. Let $p : T_f \to S^1$ be the map determined by $p(q(x,t)) = e^{2\pi i t}$. You may use the Wang sequence

$$\cdots \longrightarrow \mathrm{H}_k(S^1;A) \xrightarrow{1-f_*} \mathrm{H}_k(S^1;A) \longrightarrow \mathrm{H}_k(T_f;A) \longrightarrow \mathrm{H}_{k-1}(S^1;A) \longrightarrow \cdots.$$

In degree one, naturality identifies the map to $\mathrm{H}_0(S^1;A) \cong A$ with $p_*$ after the usual identification $\mathrm{H}_1(S^1;A) \cong A$.

(a) Extract a short exact sequence for $\mathrm{H}_1(T_f;A)$ and a kernel description of $\mathrm{H}_2(T_f;A)$. Determine $\mathrm{H}_0$ and all higher groups. Distinguish the action of $f_*$ in degrees $0$ and $1$.

(b) Choose $x \in S^1$ and a path $\gamma : [0,1] \to S^1$ from $f(x)$ to $x$. Verify that $t \mapsto q(\gamma(t), t)$ descends to a section of $p$. Use it to split the short exact sequence from part (a), without assuming that an arbitrary short exact sequence of abelian groups splits.

(c) Compute all the homology groups for $d = 1, -1, 2$, first over $\mathbb{Z}$ and then over $\mathbb{Z}/2\mathbb{Z}$. For the identity and reflection maps, identify the resulting spaces as the torus and Klein bottle. Explain which of these two spaces are distinguished by their integral homology and which are distinguished by their mod-two homology groups.