# Homotopy Groups and the Fundamental Group Action

Algebraic Topology I | Fall 2026 | Companion to Lecture 9

Use the definitions and results of Lecture 9. Concatenation $\alpha * \beta$ traverses $\alpha$ first. A path $\alpha$ from $x$ to $y$ induces transport $\tau_\alpha : \pi_n(X, x) \to \pi_n(X, y)$. There are four exercises.

## Exercise 1. Changing the Transport Path

Let $\alpha, \beta : I \to X$ run from $x$ to $y$, and let $n \geq 1$. Write $\bar{\beta}(t) = \beta(1 - t)$.

(a) Prove that $\tau_\beta^{-1} \tau_\alpha = \tau_{\alpha * \bar{\beta}}$. Deduce that transport from $x$ to $y$ is independent of the path if every loop at $x$ acts trivially on $\pi_n(X, x)$.

(b) For $n = 1$, prove that path independence holds if and only if $\pi_1(X, x)$ is abelian. Use the formula $\tau_\gamma([f]) = [\bar{\gamma} * f * \gamma]$, rather than changing the convention for concatenation.

(c) Let $q : X \to Y$ be a homotopy equivalence. Choose $x \in X$ and a homotopy inverse $r$. Explain how the trace of a homotopy $rq \simeq \mathrm{id}_X$ supplies the basepoint correction in the proof that $q_* : \pi_n(X, x) \to \pi_n(Y, q(x))$ is an isomorphism. Draw the naturality square for transport that is used in this correction.

## Exercise 2. Homotopy Groups of a Topological Group

Let $G$ be a path-connected topological group with identity $e$. Represent an element of $\pi_n(G, e)$ by a map $f : I^n \to G$ sending $\partial I^n$ to $e$.

(a) Pointwise multiplication defines a second operation on $\pi_n(G, e)$. Verify the interchange law with concatenation in the first coordinate. Deduce that $\pi_1(G, e)$ is abelian.

(b) Let $\alpha : I \to G$ be a loop at $e$. Apply the moving-boundary lemma of Lecture 9 to $H(u, t) = \alpha(t) f(u)$. Prove that $\tau_\alpha$ is the identity on $\pi_n(G, e)$ for every $n \geq 1$.

(c) Prove that the map from based homotopy classes of maps $S^n \to G$ to free homotopy classes is a bijection. State where path connectedness is used.

## Exercise 3. The First Nonzero Homotopy Group of a Product of Spheres

Let $2 \leq p \leq q$ and give $S^p \times S^q$ its product basepoint.

(a) Prove directly from cube representatives that the two projections induce an isomorphism

$$\pi_k(S^p \times S^q) \longrightarrow \pi_k(S^p) \times \pi_k(S^q)$$

for every $k \geq 1$.

(b) Compute these groups for $1 \leq k \leq p$. Treat $p = q$ separately from $p < q$.

(c) Compute $\mathrm{H}_p(S^p \times S^q; \mathbb{Z})$ from its product cell structure. Use naturality of the Hurewicz map with respect to the inclusions and projections to identify the Hurewicz isomorphism in degree $p$, including its generators.

## Exercise 4. Degree and Homotopy Equivalences

Fix $n \geq 2$ and orientations on $S^n$. You may use the connectivity of spheres, the Hurewicz theorem, and the description of free homotopy classes by basepoint transport from Lecture 9.

(a) Prove that two based self-maps are based homotopic if and only if they have the same degree. Explain why the same classification holds for free homotopy classes.

(b) Prove that a self-map $f : S^n \to S^n$ is a homotopy equivalence if and only if $\deg(f) = 1$ or $-1$. In the reverse implication, construct a homotopy inverse using degree classification, rather than assuming that a homology isomorphism is a homotopy equivalence.

(c) Let $a : S^n \to S^n$ be the antipodal map. Determine for which $n$ it is homotopic to the identity. Explain why the answer is independent of whether homotopies are required to preserve a chosen basepoint, after replacing $a$ by a based representative of its free homotopy class.