Fibrations and Homotopy Exact Sequences

Algebraic Topology I | Fall 2026 | Companion to Lecture 10

Use the homotopy exact sequence of a fibration from Lecture 10. All spaces and maps are based unless stated otherwise. There are four exercises.

Exercise 1. A Fibration with Contractible Total Space

Let $p : E \to B$ be a fibration, with fiber $F = p^{-1}(b)$ and chosen $e \in F$. Assume that E is contractible and B is path connected.

(a) For $k \geq 2$, prove that the connecting map $\pi_k(B, b) \to \pi_{k-1}(F, e)$ is an isomorphism.

(b) Show that $\pi_1(B, b) \to \pi_0(F)$ is a bijection of pointed sets. Do not call it a group isomorphism without first specifying a group structure on $\pi_0(F)$.

(c) Apply this to evaluation $P_bB \to B$ on paths beginning at b. Give an explicit contraction of P_bB, and identify its fiber. Describe the connecting map in terms of lifting a representative cube.

Exercise 2. The Hopf Map and Maps into the Two-Sphere

Let $h : S^3 \to S^2$ be the Hopf fibration with fiber S^1. You may use $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_k(S^1) = 0$ for $k \geq 2$.

(a) Write the exact sequence that proves $h_* : \pi_3(S^3) \to \pi_3(S^2)$ is an isomorphism. Identify a generator of $\pi_3(S^2)$ after choosing an orientation on S^3.

(b) If $f : S^3 \to S^3$ has degree d, determine the class of $h \circ f$ in $\pi_3(S^2)$. Prove that every based map $S^3 \to S^2$ is based homotopic to such a composite.

(c) Show that every map $S^3 \to S^2$ induces zero on reduced integral homology, but that infinitely many of these maps are not nullhomotopic. Explain why this does not contradict the Hurewicz theorem.

Exercise 3. Higher Complex Projective Spaces

For $n \geq 1$, let $p : S^{2n+1} \to \mathbb{CP}^n$ send a unit vector in $\mathbb{C}^{n+1}$ to its complex line.

(a) On the chart where the jth homogeneous coordinate is nonzero, choose the unit representative whose jth coordinate is positive real. Use it to construct a local trivialization of p with fiber S^1.

(b) Use the homotopy exact sequence to prove that $\mathbb{CP}^n$ is simply connected and compute $\pi_2(\mathbb{CP}^n)$.

(c) For $3 \leq k \leq 2n$, compute $\pi_k(\mathbb{CP}^n)$. Compute $\pi_{2n+1}(\mathbb{CP}^n)$ as well. The interval in the first sentence is empty when $n = 1$.

(d) Explain why the inclusion $\mathbb{CP}^1 \to \mathbb{CP}^n$ induces an isomorphism on π_2. Use the natural map between the two circle bundles and naturality of their connecting homomorphisms.

Exercise 4. Products and a Section of a Fibration

Let $F \to E \xrightarrow{p} B$ be a fibration of path-connected based spaces. Suppose that p admits a based section $s : B \to E$.

(a) Prove that every connecting map $\pi_k(B) \to \pi_{k-1}(F)$ is zero for $k \geq 2$.

(b) For $k \geq 2$, prove that the fiber inclusion and the section induce an isomorphism

$$\pi_k(F) \oplus \pi_k(B) \longrightarrow \pi_k(E).$$

(c) In degree one, prove instead that $\pi_1(E)$ is a semidirect product of $\pi_1(F)$ and $\pi_1(B)$. Define the action using conjugation by $s_*\pi_1(B)$, and explain why a direct product need not result.

(d) Take $F = S^1$ and let E be the mapping torus of reflection of S^1, using a fixed point to define the section over $B = S^1$. Compute the action in part (c), and give a presentation for $\pi_1(E)$.