Fundamental Groups and Attaching Maps

Algebraic Topology I | Fall 2026 | Companion to Lecture 11

Use Lecture 11 and the homology computations of Lecture 8. All displayed fundamental groups use the indicated vertex or a chosen point in the connected space. There are four exercises.

Exercise 1. A Disk Attached to a Bouquet

Let $X = S^1 \vee S^1$ with oriented loops a, b. For integers $m, n \geq 1$, attach a disk along the loop $a^m b^n$ and call the resulting cell complex $Y_{m,n}$.

(a) Use the disk-attachment form of van Kampen to give a presentation of $\pi_1(Y_{m,n})$. Describe the normal subgroup that is divided out; do not replace normal closure by the cyclic subgroup generated by the attaching word.

(b) Compute the abelianization of this presentation and compare it with a direct cellular calculation of $\mathrm{H}_1(Y_{m,n}; \mathbb{Z})$ and $\mathrm{H}_2(Y_{m,n}; \mathbb{Z})$.

(c) Show that $Y_{1,n}$ has infinite cyclic fundamental group. For $Y_{2,2}$, construct a surjection from its fundamental group onto the nonabelian group generated by two reflections of the real line. Conclude that the fundamental group of $Y_{2,2}$ is not determined by its abelianization alone.

Exercise 2. The Klein Bottle Group

Let K be the cell complex with one vertex, edges a, b, and one disk attached along $aba^{-1}b$.

(a) Give a presentation of $\pi_1(K)$. Show that every element has a representative $b^r a^s$ with $r, s \in \mathbb{Z}$.

(b) On $\mathbb{R}^2$, define $A(x, y) = (x + 1, -y)$ and $B(x, y) = (x, y + 1)$. Verify the relation $ABA^{-1} = B^{-1}$. Use this action to prove uniqueness of the normal form in part (a).

(c) Compute the center of $\pi_1(K)$ and its abelianization. Compare the latter with $\mathrm{H}_1(K; \mathbb{Z})$.

Exercise 3. Punctured Planes and Attaching Higher Cells

Choose distinct points $z_1, \dots, z_r$ in $\mathbb{R}^2$, where $r \geq 1$.

(a) Construct a graph deformation retract of $\mathbb{R}^2 \setminus \{z_1, \dots, z_r\}$, or give a van Kampen decomposition that computes its fundamental group. Specify the paths from a basepoint to the small circles around the punctures used to choose free generators.

(b) A map $S^{d-1} \to X$ attaches a d-cell to a path-connected cell complex X. For $d \geq 3$, prove that the induced map on fundamental groups is an isomorphism. State why the analogous assertion fails for $d = 2$.

(c) Starting with a bouquet of r circles, attach any collection of cells of dimensions at least 3, with the usual cell-complex topology. Prove that its fundamental group remains free on r generators. You may use that each compact subset of a cell complex is contained in a finite subcomplex. Explain how this reduces loops and homotopies to the finite case.

Exercise 4. Winding Number and Polynomial Perturbations

Let $p(z) = z^d + c_{d-1}z^{d-1} + \cdots + c_0$ be a monic complex polynomial of degree $d \geq 1$. Choose $R > 0$ so that

$$\sum_{j=0}^{d-1} |c_j| R^j < R^d.$$

(a) Prove that the loops $t \mapsto p(Re^{2\pi it})$ and $t \mapsto R^d e^{2\pi idt}$ are homotopic in $\mathbb{C} \setminus \{0\}$ by the straight-line homotopy. Compute their winding number around zero.

(b) Deduce that p has a zero in $\{z : |z| \leq R\}$. Use extension over the disk, rather than a theorem counting polynomial roots.

(c) More generally, let $f, g : \{z : |z| \leq R\} \to \mathbb{C}$ be continuous and satisfy $|f - g| < |f|$ on the boundary. Prove that their boundary loops have the same winding number. If this number is nonzero, prove that both f and g have a zero in the disk.