Covering Spaces and Free Subgroups

Algebraic Topology I | Fall 2026 | Companion to Lecture 12

Use Lecture 12. A graph is regarded as a one-dimensional cell complex. In a directed labeled covering of a bouquet, traversing an edge backward reads the inverse of its label. There are four exercises.

Exercise 1. Lifting a Circle-Valued Map

Let X be path connected and locally path connected, let $x \in X$, and let $f : X \to S^1$ satisfy $f(x) = 1$. For $m \geq 1$, let $p_m : S^1 \to S^1$ be $p_m(z) = z^m$.

(a) Prove that there is a lift $g : X \to S^1$ with $g(x) = 1$ and $p_m g = f$ if and only if $f_*\pi_1(X, x) \subseteq m\mathbb{Z}$. Explain why the based lift is unique.

(b) Prove that f has a lift to $\mathbb{R}$ under $t \mapsto e^{2\pi i t}$ if and only if $f_* = 0$. Deduce that this is also equivalent to f being nullhomotopic.

(c) Take $X = S^1$ and let f have degree d. Determine when it has an mth root as a circle-valued map, and count its distinct lifts when the value at the basepoint is not prescribed.

Exercise 2. A Three-Sheeted Cover of a Bouquet

Let $X = S^1 \vee S^1$ have oriented edges labeled a, b. Construct a labeled graph Y with vertices $0, 1, 2$. The outgoing a-edge at i ends at $i + 1$ modulo 3. The outgoing b-edges interchange 0 and 1 and fix 2. Map every edge homeomorphically onto the edge of X with the same label.

(a) Verify that the resulting map $p : Y \to X$ is a connected three-sheeted covering.

(b) Choose as a maximal tree the a-edges from 0 to 1 and from 1 to 2. For each of the four remaining oriented edges, form the loop that travels from 0 along the tree to its initial vertex, traverses the edge, and returns along the tree. Read their labels to give a free basis of $p_*\pi_1(Y, 0) \subseteq \langle a, b \rangle$.

(c) Compute the deck transformation group by finding the permutations of $\{0, 1, 2\}$ that commute with both edge permutations. Determine whether this cover is regular. Compare with the index and normalizer of the subgroup in part (b).

Exercise 3. An Infinite-Index Subgroup with Infinite Rank

Let $X = S^1 \vee S^1$ again, and let $\varphi : \langle a, b\rangle \to \mathbb{Z}$ be defined by $\varphi(a) = 1$ and $\varphi(b) = 0$. Form a labeled graph with vertices the integers, an a-edge from j to $j+1$, and a b-loop at each j.

(a) Verify that this graph is a connected covering of X. Show that the subgroup determined by the vertex 0 is $\ker\varphi$ by following the endpoint of a lifted word.

(b) Use the line of a-edges as a maximal tree. Prove that the words $a^j b a^{-j}$, for $j \in \mathbb{Z}$, freely generate $\ker\varphi$.

(c) Compute the deck group. Explain why the finite-index rank formula cannot be applied by replacing the finite index with an unspecified infinite cardinal and then performing integer subtraction.

Exercise 4. Intermediate Circle Covers

For a fixed $n \geq 1$, consider $p_n : S^1 \to S^1$. An intermediate connected cover is a factorization into covering maps

$$S^1 \xrightarrow{q} Y \xrightarrow{r} S^1, \qquad rq = p_n,$$

with chosen compatible basepoints. Intermediate covers are identified by based isomorphisms commuting with this factorization.

(a) Classify these factorizations using the subgroups $n\mathbb{Z} \subseteq H \subseteq \mathbb{Z}$. For every divisor d of n, identify q and r as power maps and compute their numbers of sheets.

(b) Compute the deck groups of all three maps and the homomorphism between deck groups associated with each intermediate cover. Describe its kernel.

(c) For $n = 12$, draw the subgroup-inclusion diagram and the corresponding diagram of intermediate covers. Mark each covering arrow with its number of sheets, and explain why a smaller subgroup corresponds to a cover with more sheets.