# Modules, Derived Categories, and the Tensor Product

Algebraic Topology I | Fall 2026 | September 9

## 2.1 Exact Sequences and Mapping Cones

Let $\mathcal{A}$ be a locally small abelian category. We use homological grading.

A concrete example is $\mathcal{A} = \mathrm{Mod}_R$ for a commutative ring $R$. This is an abelian category: for a module map $f : M \to N$, its kernel is the usual submodule $\ker f \subseteq M$ and its cokernel is $N/\operatorname{im} f$. For an ideal $I \subseteq R$, the inclusion and quotient maps give the short exact sequence $0 \to I \to R \to R/I \to 0$, also viewed as a sequence of complexes concentrated in degree zero.

Recall: Use the chain complexes and chain maps of Lecture 1, Definition 1.3.3, and the cycle objects, boundary objects, homology, and acyclicity of Definition 1.3.4. Write $\mathrm{Ch}(\mathcal{A})$ for their category. A complex is *acyclic* if its homology is zero.

**Definition 2.1.1.** A chain map is a *quasi-isomorphism* if every induced homology map is an isomorphism.

### 2.1.1 The Snake Lemma

**Lemma 2.1.2.** Consider a commutative diagram in an abelian category with exact rows:

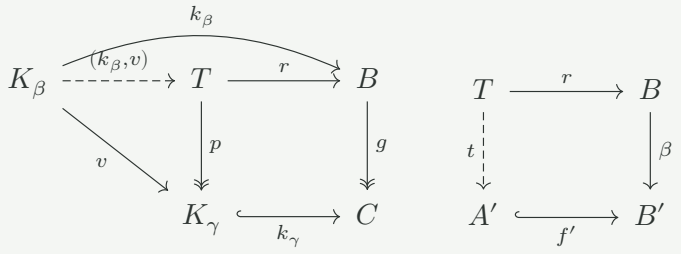
$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 & & \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C'. & & \end{array}$$

There is a natural exact sequence

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma.$$

If $f$ is monic, the sequence starts with $0$; if $g'$ is epic, it ends with $0$.

**Proof.** Write $k_\eta : K_\eta \hookrightarrow \mathrm{dom}(\eta)$ and $q_\eta : \mathrm{cod}(\eta) \twoheadrightarrow Q_\eta$ for the kernel and cokernel of $\eta \in \{\alpha, \beta, \gamma\}$. Denote the induced maps by $u : K_\alpha \to K_\beta$, $v : K_\beta \to K_\gamma$, $\bar{f} : Q_\alpha \to Q_\beta$, and $\bar{g} : Q_\beta \to Q_\gamma$. Form the pullback $T = B \times_C K_\gamma$. The first square makes $p$ epic; since $g'\beta r = \gamma k_\gamma p = 0$, the second defines $t$ uniquely.



The map $j = (f, 0) : A \to T$ has image $\ker p$. Since $tj = \alpha$, the cokernel property of $p$ gives the unique dashed arrow below:

$$\begin{array}{ccccccc} A & \xrightarrow{j} & T & \xrightarrow{p} & K_\gamma & \longrightarrow & 0 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle\delta} & & \\ A' & = & A' & \xrightarrow[q_\alpha]{} & Q_\alpha & \longrightarrow & 0. \end{array}$$

Thus $\delta p = q_\alpha t$. The lift in the first picture satisfies $t(k_\beta, v) = 0$, so $\delta v = 0$. Moreover, $\bar{f}\delta p = q_\beta \beta r = 0$, hence $\bar{f}\delta = 0$ because $p$ is epic. The other composites $vu$ and $\bar{g}\bar{f}$ vanish by exactness of the rows.

To test whether a map $x : X \to Y$ factors through an image, it suffices to lift $xe$ through the original map after an epimorphism $e : E \twoheadrightarrow X$. Such lifts are obtained by pullback along the epimorphism onto that image; epimorphisms detect factorization through subobjects. In each panel below, $e$ may be refined by these pullbacks, and every dashed lift has source $E$.

*Exactness at $K_\beta$.* For $x : X \to K_\beta$ with $vx = 0$, lift $k_\beta x e$ through $f$.

$$\begin{array}{ccccccc} & & \overset{\lambda}{\frown} & & & & \\ E & \overset{\tilde{\lambda}}{\dashrightarrow} & K_\alpha & \overset{k_\alpha}{\hookrightarrow} & A & \xrightarrow{\alpha} & A' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ X & \xrightarrow[x]{} & K_\beta & \xhookrightarrow[k_\beta]{} & B & \xrightarrow[\beta]{} & B'. \end{array}$$

Indeed, $f'\alpha\lambda = \beta k_\beta x e = 0$ gives $\alpha\lambda = 0$, so the kernel property supplies $\tilde{\lambda}$ and $u\tilde{\lambda} = xe$.

*Exactness at $K_\gamma$.* For $x : X \to K_\gamma$ with $\delta x = 0$, the lifts are

$$\begin{array}{ccccc} E & \overset{s}{\dashrightarrow} & T & \xrightarrow{t} & A' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q_\alpha} \\ X & \xrightarrow[x]{} & K_\gamma & \xrightarrow[\delta]{} & Q_\alpha \end{array} \qquad \begin{array}{ccc} E & \overset{\lambda}{\dashrightarrow} & A \\ {\scriptstyle ts}\downarrow & & \downarrow{\scriptstyle\alpha} \\ A' & = & A' \end{array} \qquad \begin{array}{ccc} E & \overset{h}{\dashrightarrow} & K_\beta \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow[x]{} & K_\gamma. \end{array}$$

Here $q_\alpha ts = 0$ permits $\lambda$, and the corrected lift $h$ is defined by

$$k_\beta h = rs - f\lambda, \qquad \beta(rs - f\lambda) = f'(ts - \alpha\lambda) = 0, \qquad g(rs - f\lambda) = k_\gamma x e.$$

*Exactness at $Q_\alpha$.* For $x : X \to Q_\alpha$ with $\bar{f}x = 0$, lift first through $q_\alpha$ and then through $\beta$:

$$\begin{array}{ccccc} E & \overset{a}{\dashrightarrow} & A' & \xrightarrow{f'} & B' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle q_\alpha} & & \downarrow{\scriptstyle q_\beta} \\ X & \xrightarrow[x]{} & Q_\alpha & \xrightarrow[\bar{f}]{} & Q_\beta \end{array} \qquad \begin{array}{ccc} E & \overset{b}{\dashrightarrow} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle\beta} \\ A' & \xhookrightarrow[f']{} & B' \end{array} \qquad \begin{array}{ccccc} & & \overset{\nu}{\frown} & & \\ E & \overset{(b,\nu)}{\dashrightarrow} & T & \xrightarrow{p} & K_\gamma \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle q_\alpha t} & & \downarrow{\scriptstyle\delta} \\ X & \xrightarrow[x]{} & Q_\alpha & = & Q_\alpha. \end{array}$$

The kernel property defines $\nu$ by $k_\gamma \nu = gb$, since $\gamma g b = g'\beta b = g' f' a = 0$. The pullback lift $(b, \nu) : E \to T$ has $t(b, \nu) = a$, hence $\delta\nu = q_\alpha a = xe$.

*Exactness at $Q_\beta$.* For $x : X \to Q_\beta$ with $\bar{g}x = 0$, lift successively through $q_\beta$, $\gamma$, and $g$:

$$\begin{array}{ccccc} E & \overset{b'}{\dashrightarrow} & B' & \xrightarrow{g'} & C' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle q_\beta} & & \downarrow{\scriptstyle q_\gamma} \\ X & \xrightarrow[x]{} & Q_\beta & \xrightarrow[\bar{g}]{} & Q_\gamma \end{array} \qquad \begin{array}{ccc} E & \overset{b}{\dashrightarrow} & B \\ {\scriptstyle c}\vdots & & \downarrow{\scriptstyle g} \\ C & = & C \end{array} \qquad \begin{array}{ccc} E & \overset{a'}{\dashrightarrow} & A' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle q_\beta f'} \\ X & \xrightarrow[x]{} & Q_\beta. \end{array}$$

These lifts satisfy $\gamma c = g'b'$ and $gb = c$. The corrected lift $a'$ is the unique map with

$$f'a' = b' - \beta b, \qquad g'(b' - \beta b) = \gamma(c - gb) = 0, \qquad \bar{f} q_\alpha a' = q_\beta(b' - \beta b) = xe.$$

The four panels prove the reverse image–kernel inclusions. If $f$ is monic, then $k_\beta u = f k_\alpha$ makes $u$ monic. If $g'$ is epic, then $\bar{g} q_\beta = q_\gamma g'$ makes $\bar{g}$ epic. Finally, a morphism of the original row diagrams induces a morphism of their pullbacks $T$; uniqueness of $t$ and $\delta$ makes the induced square on connecting maps commute. □

For modules the construction is the familiar rule: lift $c \in \ker \gamma$ to $b \in B$, solve $f'a' = \beta b$, and put $\delta(c) = [a']$. Changing $b$ to $b + fa$ changes $a'$ to $a' + \alpha a$, so the cokernel class does not change.

### 2.1.2 The Homology Long Exact Sequence

**Theorem 2.1.3.** A degreewise short exact sequence of complexes

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\rho} C \longrightarrow 0 \tag{1}$$

induces natural connecting morphisms $\delta_n : \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ and a long exact sequence. Five consecutive terms are

$$\mathrm{H}_n(A) \xrightarrow{\mathrm{H}_n(\iota)} \mathrm{H}_n(B) \xrightarrow{\mathrm{H}_n(\rho)} \mathrm{H}_n(C) \xrightarrow{\delta_n} \mathrm{H}_{n-1}(A) \xrightarrow{\mathrm{H}_{n-1}(\iota)} \mathrm{H}_{n-1}(B). \tag{2}$$

For modules, if $\zeta \in C_n$ is a cycle, choose $\beta \in B_n$ with $\rho\beta = \zeta$ and write $\partial^B \beta = \iota\alpha$. Then $\alpha$ is a cycle and $\delta_n$ sends the class of $\zeta$ to the class of $\alpha$.

**Proof.** Put $Q_n(X) = \operatorname{coker}(\partial^X_{n+1}) = X_n / B_n(X)$. The differential induces $\partial_X : Q_n(X) \to Z_{n-1}(X)$. Its kernel is $\mathrm{H}_n(X)$ and its cokernel is $\mathrm{H}_{n-1}(X)$: its image is precisely $B_{n-1}(X)$.

The following commutative diagram has exact rows:

$$\begin{array}{ccccccccc} & & Q_n(A) & \longrightarrow & Q_n(B) & \longrightarrow & Q_n(C) & \longrightarrow & 0 \\ & & {\scriptstyle \partial_A}\downarrow & & \downarrow{\scriptstyle \partial_B} & & \downarrow{\scriptstyle \partial_C} & & \\ 0 & \longrightarrow & Z_{n-1}(A) & \longrightarrow & Z_{n-1}(B) & \longrightarrow & Z_{n-1}(C). & & \end{array} \tag{3}$$

The upper row is exact because cokernels preserve right exact sequences of arrows, applied to the differentials in degrees $n+1, n$. The lower row is exact because kernels preserve left exact sequences of arrows, applied in degrees $n-1, n-2$.

Lemma 2.1.2, in its form with a right exact upper row and a left exact lower row, gives

$$\ker \partial_A \to \ker \partial_B \to \ker \partial_C \xrightarrow{\delta_n} \operatorname{coker} \partial_A \to \operatorname{coker} \partial_B \to \operatorname{coker} \partial_C.$$

Substituting the identifications above proves exactness at all three types of homology terms. The six-term sequences for consecutive $n$ join because their other maps are the maps induced

by $\iota, \rho$. Naturality follows from the snake lemma; see [Stacks, Tag 0117, Long Exact Cohomology Sequence].

For modules, $\rho\partial^B\beta = \partial^C\zeta = 0$, so exactness gives a unique $\alpha$ with $\iota\alpha = \partial^B\beta$. Since $\iota(\partial^A\alpha) = \partial^B\partial^B\beta = 0$ and $\iota$ is mono, $\partial^A\alpha = 0$. Changing $\beta$ to $\beta + \iota\alpha'$ changes $\alpha$ to $\alpha + \partial^A\alpha'$. Changing $\zeta$ to $\zeta + \partial^C\zeta'$ and choosing a lift $\beta'$ of $\zeta'$ changes a chosen lift to $\beta + \partial^B\beta'$ and leaves its differential unchanged. Thus the formula is well defined and agrees with the snake map. □

**Example 2.1.4.** If $A \subseteq B$ is a subcomplex, define the quotient complex $B/A$ degreewise. The theorem gives a long exact sequence with five consecutive terms

$$\mathrm{H}_n(A) \to \mathrm{H}_n(B) \to \mathrm{H}_n(B/A) \xrightarrow{\delta_n} \mathrm{H}_{n-1}(A) \to \mathrm{H}_{n-1}(B).$$

For singular chains of a subspace inclusion $X' \subseteq X$, take $B = \mathrm{C}_*(X; R)$ and $A = \mathrm{C}_*(X'; R)$. A relative cycle is represented by a chain in $X$ whose boundary lies in $X'$; $\delta$ takes the homology class of that boundary in $X'$.

### 2.1.3 The Cone Sequence

For cones and termwise split sequences, see [Stacks, Tag 014D]. The Stacks Project uses cohomological grading; these notes use homological grading.

**Definition 2.1.5.** For $r \in \mathbb{Z}$, set $A[r]_n = A_{n-r}$ and $\partial_n^{A[r]} = (-1)^r\partial_{n-r}^A$. A chain map shifts by $f[r]_n = f_{n-r}$. Thus $\mathrm{H}_n(A[r]) = \mathrm{H}_{n-r}(A)$ canonically. For any biproduct, $\iota_j$ and $\rho_j$ denote the canonical inclusions and projections.

**Definition 2.1.6.** For a chain map $f : A \to B$, define

$$\mathrm{Cone}(f)_n = B_n \oplus A_{n-1}, \qquad \partial_n^{\mathrm{Cone}(f)} = \begin{pmatrix} \partial_n^B & f_{n-1} \\ 0 & -\partial_{n-1}^A \end{pmatrix}.$$

The upper-right entry of its squared differential is $\partial_{n-1}^B f_{n-1} - f_{n-2}\partial_{n-1}^A = 0$; all other entries vanish. There is a degreewise split short exact sequence

$$0 \to B \xrightarrow{\jmath=\iota_1} \mathrm{Cone}(f) \xrightarrow{q=\rho_2} A[1] \to 0. \tag{4}$$

**Proposition 2.1.7.** The connecting map of (4) is $\mathrm{H}_{n-1}(f)$. Hence

$$\cdots \to \mathrm{H}_n(A) \xrightarrow{\mathrm{H}_n(f)} \mathrm{H}_n(B) \xrightarrow{\mathrm{H}_n(\jmath)} \mathrm{H}_n(\mathrm{Cone}(f)) \xrightarrow{\mathrm{H}_n(q)} \mathrm{H}_{n-1}(A) \xrightarrow{\mathrm{H}_{n-1}(f)} \mathrm{H}_{n-1}(B) \to \cdots \tag{5}$$

is exact. In particular, $f$ is a quasi-isomorphism if and only if $\mathrm{Cone}(f)$ is acyclic.

**Proof.** A cycle $\zeta \in A_{n-1}$ lifts to $(0, \zeta)$, whose cone differential is $(f(\zeta), 0)$. Thus the connecting map has the asserted positive sign. Categorically, the second-summand section of (4) has the same differential defect $f$, so this computes the snake map in $\mathcal{A}$. Apply Theorem 2.1.3. If every $\mathrm{H}_n(f)$ is invertible, exactness forces all cone homology to vanish. Conversely, its vanishing makes $\mathrm{H}_n(f)$ both mono and epi, hence an isomorphism in $\mathcal{A}$. □

**Example 2.1.8.** For $A = B = \mathbb{Z}$ concentrated in degree zero and $f = m$, the cone is $\mathbb{Z} \xrightarrow{m} \mathbb{Z}$ in degrees $1, 0$. Its homology is $\ker(m)$ in degree $1$ and $\mathbb{Z}/m\mathbb{Z}$ in degree $0$.

More generally, for a commutative ring $R$ and $r \in R$, the cone of multiplication by $r$ on $R$ is $R \xrightarrow{r} R$ in degrees $1, 0$. Its homology is $\{a \in R : ra = 0\}$ in degree $1$ and $R/rR$ in degree $0$, and is zero in all other degrees. If $r$ is a non-zero-divisor, the quotient map from this cone to $R/rR$ concentrated in degree zero is a quasi-isomorphism.

### 2.1.4 Recovering a Short Exact Sequence from a Cone

**Proposition 2.1.9.** For (1), let $\pi : \mathrm{Cone}(\iota) \to C$ have restrictions $\rho : B \to C$ and $0 : A[1] \to C$. Then $\pi$ is a quasi-isomorphism and the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\iota} & B & \xrightarrow{j} & \mathrm{Cone}(\iota) \\ \| & & \| & & \sim\downarrow\pi \\ A & \xrightarrow[\iota]{} & B & \xrightarrow[\rho]{} & C \end{array}$$

commutes. With $q : \mathrm{Cone}(\iota) \to A[1]$ the second projection, the connecting morphism in Theorem 2.1.3 is

$$\delta_n = -\mathrm{H}_n(q)\,\mathrm{H}_n(\pi)^{-1}. \tag{6}$$

**Proof.** The map $\pi$ is a degreewise epimorphism of complexes. Its kernel identifies with $\mathrm{Cone}(\mathrm{id}_A)$: the inclusion sends $(\alpha', \alpha)$ to $(\iota\alpha', \alpha)$. On this kernel, $s_n(\alpha', \alpha) = (0, \alpha')$ is a contracting homotopy, since $\partial s + s\partial = \mathrm{id}$. Theorem 2.1.3 therefore implies that $\pi$ is a quasi-isomorphism.

For the sign, lift a cycle $\zeta$ to $\beta$ and write $\partial^B\beta = \iota\alpha$. The corresponding cone cycle is $(\beta, -\alpha)$, so $q(\beta, -\alpha) = -\alpha$, while $\delta_n$ sends the class of $\zeta$ to the class of $\alpha$. The same computation with the biproduct maps gives the identity in $\mathcal{A}$. □

Thus the cone sequence recovers the long exact sequence of (1); formula (6) identifies the chosen connecting map, including its sign.

## 2.2 Homotopy and Derived Categories

### 2.2.1 Passing to Chain Homotopy Classes

Use chain homotopies and contractible complexes as in Lecture 1, Definition 1.3.5. The new construction here is the quotient by chain homotopy.

**Definition 2.2.1.** Let $N(A, B)$ be the subgroup of nullhomotopic chain maps. The category $K(\mathcal{A})$ has the same objects as $\mathrm{Ch}(\mathcal{A})$ and morphism groups

$$\mathrm{Mor}_{K(\mathcal{A})}(A, B) = \mathrm{Mor}_{\mathrm{Ch}(\mathcal{A})}(A, B)/N(A, B).$$

Composition is induced by composition of chain maps; homotopy is as above.

**Proposition 2.2.2.** Nullhomotopic maps form an additive ideal. Hence $K(\mathcal{A})$ is additive and every homology functor factors through it. Its isomorphisms are precisely chain homotopy equivalences.

**Proof.** The sum of homotopies is a homotopy for the sum. If $f = \partial h + h\partial$ and $\alpha, \beta$ are composable chain maps, then $\beta f \alpha = \partial(\beta h\alpha) + (\beta h\alpha)\partial$. Thus composition descends to the quotient. Lecture 1, Proposition 1.3.6, supplies equality of the induced homology maps. An inverse in the quotient means exactly inverse chain maps up to homotopy. □

**Example 2.2.3.** The acyclic, noncontractible complex of Lecture 1, following Proposition 1.3.6, is therefore a nonzero object of $K(\mathbf{Ab})$.

### 2.2.2 Inverting Quasi-Isomorphisms

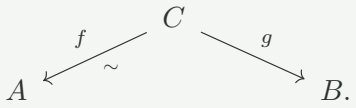
Assume here that $\mathcal{A}$ is essentially small or is a category of modules. The discussion of the size issue begins in Subsection 2.2.3. For a complementary account of the passage from complexes to the homotopy and derived categories, see Weibel [Wei94, §§10.1,10.3].

**Definition 2.2.4.** Let $S$ be the class of quasi-isomorphisms in $K(\mathcal{A})$. Define

$$D(\mathcal{A}) = K(\mathcal{A})[S^{-1}].$$

It has the same objects as $K(\mathcal{A})$ and a functor $Q : K(\mathcal{A}) \to D(\mathcal{A})$ sending each $s \in S$ to an isomorphism. Its universal property is that precomposition with $Q$ identifies functors out of $D(\mathcal{A})$ with functors out of $K(\mathcal{A})$ which invert $S$; natural transformations correspond as well.

Morphisms are represented by spans in $K(\mathcal{A})$:




The represented map is $Q(g)Q(f)^{-1}$.

Here $\sim$ marks a quasi-isomorphism: $f$ induces an isomorphism in every homology degree, but need not be an isomorphism of complexes. Two spans are equivalent if they admit a common refinement in $K(\mathcal{A})$ on which both maps agree, with the common map to $A$ a quasi-isomorphism. This is the standard calculus-of-fractions description for quasi-isomorphisms; we use it here without proof.

Homology descends to $D(\mathcal{A})$ by the universal property. A complex becomes zero precisely when it is acyclic: use the quasi-isomorphism $0 \to A$ in one direction, and homology in the other. Similarly, a chain map becomes invertible precisely when it is a quasi-isomorphism.

**Definition 2.2.5.** The quasi-isomorphism $\pi : \mathrm{Cone}(\iota) \to C$ gives a morphism

$$\delta = Q(-q)Q(\pi)^{-1} : C \longrightarrow A[1]$$

in $D(\mathcal{A})$, with $\mathrm{H}_n(\delta) = \delta_n$. Thus the short exact sequence determines

$$A \xrightarrow{\iota} B \xrightarrow{\rho} C \xrightarrow{\delta} A[1].$$

This is its distinguished triangle. With our signs, cone triangles have the form $X \xrightarrow{f} Y \xrightarrow{\jmath} \mathrm{Cone}(f) \xrightarrow{-q} X[1]$.

### 2.2.3 Complexes and Essentially Small Categories

**Proposition 2.2.6.** If $\mathcal{A}$ is locally small, then $\mathrm{Ch}(\mathcal{A})$ and $K(\mathcal{A})$ are locally small, even for unbounded complexes.

**Proof.** For fixed $A, B$, every chain map lies in the set

$$\prod_{n \in \mathbb{Z}} \mathrm{Mor}_{\mathcal{A}}(A_n, B_n).$$

The chain-map equations select a subset. The nullhomotopic maps form a subgroup of this set-sized abelian group; quotienting by it gives a set. The countable product here is a product of sets, not a product that must exist in $\mathcal{A}$. □

**Proposition 2.2.7.** If $\mathcal{A}$ is essentially small, then $D(\mathcal{A})$ is locally small.

**Proof.** Replace $\mathcal{A}$ by a small skeleton. Its complexes and chain maps form sets. Finite words in arrows and formal inverses also form a set; localization quotients this set by relations. □

### 2.2.4 Local Smallness of Derived Module Categories

**Proposition 2.2.8.** For a set-sized ring $R$, the unbounded category $D(\mathrm{Mod}_R)$ is locally small.

**Proof.** Fix complexes $A, B$, and an infinite cardinal $\kappa$ at least $|R|$, $\aleph_0$, and the cardinality of the disjoint union of the underlying sets $A_n$. Consider any span $A \xleftarrow{f} C \xrightarrow{g} B$ with $f$ a quasi-isomorphism.

Choose cycles in $C$ representing every homology class of $A$. They generate a subcomplex $C^{(0)}$ of total cardinality at most $\kappa$. From $C^{(m)}$, form $C^{(m+1)}$ by adjoining one bounding element in $C$ for each cycle whose image bounds in $A$, then closing under the differential and module operations. These bounds exist since $\mathrm{H}_*(f)$ is injective; the cardinality remains at most $\kappa$.

The union $C' = \bigcup_{m\geq 0} C^{(m)}$ still has size at most $\kappa$, and $f|_{C'}$ is a quasi-isomorphism: chosen cycles give surjectivity, and the successive bounds give injectivity. Thus every span refines to one whose middle complex has size at most $\kappa$. There is a set of such complexes up to isomorphism and a set of maps from each to $A, B$. Quotienting this set of spans gives the required morphism set. □

**Remark 2.2.9.** For background on size questions beyond the module case, see [CN08]. General treatments using injective replacements appear in [Stacks, Tag 070G and Tag 079P].

## 2.3 Tensor Products and Their Computation

### 2.3.1 The Universal Bilinear Map

All rings have an identity and all modules are unital right modules; unless specified, modules and presentations need not be finitely generated. Write $\mathrm{Mod}_R$ for the category of right $R$-modules and $\mathrm{Mor}_{\mathrm{Mod}_R}(A, B)$ for its morphism groups. Fix a commutative ring $R$. Every right $R$-module then has the canonical left action $r \cdot n = nr$, which is used on the second factor of a tensor product. An $R$-bilinear map $\beta : A \times B \to C$ is linear in each variable separately. The tensor product turns such a map into a linear map with a single input.

**Definition 2.3.1.** The tensor product of $A$ and $B$ is an $R$-module $A \otimes_R B$ with a bilinear map $\tau : A \times B \to A \otimes_R B$ such that, for every bilinear $\beta : A \times B \to C$, there is a unique linear $\tilde{\beta}$ with $\tilde{\beta}\tau = \beta$. Write $\tau(m, n) = m \otimes n$.

$$\begin{array}{ccc} A \times B & \xrightarrow{\ \beta\ } & C \\ {\scriptstyle\tau}\downarrow & \nearrow_{\tilde{\beta}} & \\ A \otimes_R B & & \end{array}$$

**Proposition 2.3.2.** Choose arbitrary free resolutions

$$\cdots \longrightarrow P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\epsilon} A \longrightarrow 0, \qquad \cdots \longrightarrow Q_1 \xrightarrow{\partial_1^Q} Q_0 \xrightarrow{\eta} B \longrightarrow 0.$$

Here a free resolution means an exact augmented sequence of free $R$-modules. Choose bases $P_a = \bigoplus_{\lambda \in \Lambda_a} R$ and $Q_a = \bigoplus_{\gamma \in \Gamma_a} R$ for $a = 0, 1$, and write $\partial_1^P(p_{\lambda'}) = \sum_{\lambda \in \Lambda_0} p_\lambda a_{\lambda\lambda'}$ for $\lambda' \in \Lambda_1$ and $\partial_1^Q(q_{\gamma'}) = \sum_{\gamma \in \Gamma_0} q_\gamma b_{\gamma\gamma'}$ for $\gamma' \in \Gamma_1$. There is a natural coequalizer formula in $R$-modules

$$A \otimes_R B \cong \mathrm{coeq}\left( \left( \bigoplus_{\Lambda_1 \times \Gamma_0} R \right) \oplus \left( \bigoplus_{\Lambda_0 \times \Gamma_1} R \right) \underset{0}{\overset{D}{\rightrightarrows}} \bigoplus_{\Lambda_0 \times \Gamma_0} R \right),$$

where, writing $[\lambda,\gamma]$ for the target basis,

$$D([\lambda',\gamma],0)=\sum_{\lambda}[\lambda,\gamma]a_{\lambda\lambda'},\qquad D(0,[\lambda,\gamma'])=\sum_{\gamma}[\lambda,\gamma]b_{\gamma\gamma'}.$$

Thus $D$ is the map induced by the two degree-one differentials, $\partial_1^P\otimes\mathrm{id}+\mathrm{id}\otimes\partial_1^Q$, under the standard identification of tensor products of free modules with free modules on pairs of basis elements.

**Proof.** A linear map from $\bigoplus_{\Lambda_0\times\Gamma_0}R$ to an $R$-module $C$ is the same as an $R$-bilinear map $\beta : P_0\times Q_0\to C$: specify its values on basis pairs and extend in both variables. Such a linear map coequalizes $D$ and zero exactly when $\beta(\partial_1^P a,q)=0$ for all $a\in P_1$, $q\in Q_0$, and $\beta(p,\partial_1^Q b)=0$ for all $p\in P_0$, $b\in Q_1$. Equivalently, $\beta$ vanishes on $\operatorname{im}\partial_1^P\times Q_0$ and on $P_0\times\operatorname{im}\partial_1^Q$. Since $A=P_0/\operatorname{im}\partial_1^P$ and $B=Q_0/\operatorname{im}\partial_1^Q$, these conditions say precisely that $\beta$ descends uniquely to an $R$-bilinear map $A\times B\to C$. The coequalizer therefore has the universal property in Definition 2.3.1. In a module category the coequalizer of $D$ and zero is the quotient by $\operatorname{im}D$. The universal property also gives uniqueness up to a unique isomorphism compatible with the universal bilinear map. □

**Corollary 2.3.3.** In other words, with the free resolutions and bases of Proposition 2.3.2, $A\otimes_R B$ is the free module on symbols $[\lambda,\gamma]$, for $(\lambda,\gamma)\in\Lambda_0\times\Gamma_0$, modulo the relations

$$\sum_{\lambda}[\lambda,\gamma]a_{\lambda\lambda'}=0\quad(\lambda'\in\Lambda_1,\ \gamma\in\Gamma_0),\qquad\sum_{\gamma}[\lambda,\gamma]b_{\gamma\gamma'}=0\quad(\lambda\in\Lambda_0,\ \gamma'\in\Gamma_1).$$

The class of $[\lambda,\gamma]$ represents $\epsilon(p_\lambda)\otimes\eta(q_\gamma)$. All sums are finite because each differential sends a basis element to a finite linear combination. Only degrees zero and one of the chosen resolutions enter this formula.

**Proof.** The coequalizer of $D$ and zero is $\operatorname{coker}D$. The image of $D$ is generated by its values on the basis elements of its two source summands, which are exactly the displayed relations. □

Every tensor is a finite sum of elementary tensors. Such an expression need not be unique, and a tensor need not be elementary. For example, over a field $k$, the tensor $e_1\otimes e_1+e_2\otimes e_2$ in $k^2\otimes_k k^2$ corresponds to a rank-two matrix, whereas an elementary tensor corresponds to a matrix of rank at most one.

Linear maps $f : A\to A'$ and $g : B\to B'$ induce $f\otimes g : A\otimes_R B\to A'\otimes_R B'$ by $m\otimes n\mapsto f(m)\otimes g(n)$. The universal property also gives the natural isomorphisms

$$R\otimes_R B\cong B,\qquad A\otimes_R B\cong B\otimes_R A,\qquad\left(\bigoplus_{\lambda\in\Lambda}R\right)\otimes_R B\cong\bigoplus_{\lambda\in\Lambda}B.$$

The first map sends $r\otimes n$ to $nr$, the second interchanges the factors, and the third sends $e_\lambda\otimes n$ to $n$ in the summand indexed by $\lambda$. Their inverse maps are specified on the same generators. Each vector in a direct sum has finite support. For a finite indexing set $\Lambda$, we write $R^\Lambda$ and $B^\Lambda$.

**Proposition 2.3.4.** There is a natural isomorphism

$$\mathrm{Mor}_{\mathrm{Mod}_R}(A\otimes_R B,C)\cong\mathrm{Mor}_{\mathrm{Mod}_R}(A,\mathrm{Mor}_{\mathrm{Mod}_R}(B,C)).$$

For fixed $B$, tensoring preserves cokernels and direct sums. In particular, if $A'\xrightarrow{u}A\xrightarrow{v}A''\to 0$ is exact, then

$$A'\otimes_R B\xrightarrow{u\otimes\mathrm{id}}A\otimes_R B\xrightarrow{v\otimes\mathrm{id}}A''\otimes_R B\longrightarrow 0$$

is exact.

**Proof.** Commutativity makes $\mathrm{Mor}_{\mathrm{Mod}_R}(B,C)$ an $R$-module by $(\phi r)(n)=\phi(n)r$. A bilinear map $\beta : A\times B\to C$ determines the linear map $\tilde{\beta} : A\to \mathrm{Mor}_{\mathrm{Mod}_R}(B,C)$ specified by $\tilde{\beta}(m)(n)=\beta(m,n)$. This correspondence identifies bilinear maps with the right-hand side. The universal property identifies them with the left-hand side. Thus $-\otimes_R B$ is a left adjoint, so it preserves colimits, including cokernels and direct sums. Explicitly, a bilinear map on $A\times B$ vanishing on $u(A')\times B$ factors uniquely through $A''\times B$; this is exactly the asserted cokernel property. □

### 2.3.2 Tensor Products from Presentations

A presentation is an exact sequence $\bigoplus_{\gamma\in\Gamma} R \xrightarrow{\phi} \bigoplus_{\lambda\in\Lambda} R \to A\to 0$. Its basis elements indexed by $\Lambda$ give generators $m_\lambda$ of $A$, and its columns give relations among those generators. Such a presentation always exists: take a free module surjecting onto $A$, then a free module surjecting onto the kernel.

**Proposition 2.3.5.** For an arbitrary presentation of $A$ and an arbitrary module $B$,

$$A\otimes_R B \cong \mathrm{coker}(\phi_B : \bigoplus_{\gamma\in\Gamma} B \longrightarrow \bigoplus_{\lambda\in\Lambda} B). \tag{7}$$

If $\phi(e_\gamma)=\sum_\lambda e_\lambda a_{\lambda\gamma}$, then

$$\phi_B((n_\gamma)_{\gamma\in\Gamma}) = \left(\sum_{\gamma\in\Gamma} n_\gamma a_{\lambda\gamma}\right)_{\lambda\in\Lambda}.$$

Each column of $\phi$ has finite support, and each input has finite support, so this formula is defined even for infinite sets.

If also $\bigoplus_{\gamma'\in\Gamma'} R \xrightarrow{\psi} \bigoplus_{\lambda'\in\Lambda'} R \to B\to 0$ is a presentation, then

$$\begin{aligned}&\left(\bigoplus_{(\gamma,\lambda')\in\Gamma\times\Lambda'} R\right)\oplus\left(\bigoplus_{(\lambda,\gamma')\in\Lambda\times\Gamma'} R\right)\\ &\xrightarrow{(\phi\otimes\mathrm{id}\ \ \mathrm{id}\otimes\psi)} \bigoplus_{(\lambda,\lambda')\in\Lambda\times\Lambda'} R \longrightarrow A\otimes_R B\longrightarrow 0.\end{aligned} \tag{8}$$

This sequence is exact. The first arrow applies $\phi\otimes\mathrm{id}$ to the first summand and $\mathrm{id}\otimes\psi$ to the second, then adds their images in the free module on $\Lambda\times\Lambda'$.

**Proof.** Apply Proposition 2.3.4 to the presentation of $A$ and identify the tensors of free modules with direct sums of $B$. This proves (7). For (8), the generator indexed by $(\lambda,\lambda')$ maps to $m_\lambda\otimes n_{\lambda'}$. The relations from $\phi$ impose every relation among the $m_\lambda$ for each fixed $n_{\lambda'}$, and those from $\psi$ impose every relation among the $n_{\lambda'}$ for each fixed $m_\lambda$. A linear map out of this cokernel is therefore precisely a bilinear map $A\times B\to C$. Apply Definition 2.3.1. □

**Remark 2.3.6.** For finite presentations $\phi : R^a\to R^b$ and $\psi : R^c\to R^d$, formula (8) is the cokernel of the block matrix

$$\left[\,\phi\otimes I_d \;\middle|\; I_b\otimes\psi\,\right] : R^{ad+bc}\longrightarrow R^{bd},$$

using the basis order specified by the tensor factors. Over $\mathbb{Z}$ one can reduce finite matrices by Smith normal form. Over an arbitrary ring, the cokernel presentation is still a complete algebraic description, but a finite effective algorithm for simplifying it requires additional input about the ring. Arbitrary modules need not admit finite presentations.

**Example 2.3.7.** For any ideal $I\subseteq R$,

$$(R/I)\otimes_R B\cong B/IB.$$

The map sends $(r+I)\otimes n$ to $nr+IB$, with inverse induced by the map $B\to (R/I)\otimes_R B$ given by $n\mapsto (1+I)\otimes n$. Here $IB$ consists of finite sums of elements $an=na$ with $a\in I$ and $n\in B$. If $B$ is a commutative $R$-algebra, $IB$ is the ideal generated by the image of $I$. Hence

$$(R/I)\otimes_R (R/J)\cong R/(I+J).$$

For positive integers $a,b$, this gives $\mathbb{Z}/a\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/b\mathbb{Z}\cong \mathbb{Z}/\gcd(a,b)\mathbb{Z}$. In particular, nonzero modules can have zero tensor product.

**Example 2.3.8.** Let $A=\operatorname{coker}\phi$ over $\mathbb{Z}$, where

$$\phi=\begin{pmatrix}2&0\\4&6\end{pmatrix}:\mathbb{Z}^2\longrightarrow\mathbb{Z}^2,\qquad B=\mathbb{Z}/4\mathbb{Z}.$$

Tensoring replaces $\phi$ by its action on $B^2$, which is the diagonal matrix with entries $2,2$. Therefore

$$A\otimes_{\mathbb{Z}} B\cong B^2/(2B\oplus 2B)\cong(\mathbb{Z}/2\mathbb{Z})^2.$$

In coordinates, the map on $(\mathbb{Z}/4\mathbb{Z})^2$ is

$$\phi_B(a,b)=(2a,4a+6b)=(2a,2b).$$

Its image consists of $(0,0),(2,0),(0,2),(2,2)$. The map $(\mathbb{Z}/4\mathbb{Z})^2\to(\mathbb{Z}/2\mathbb{Z})^2$ given by $(a,b)\mapsto(a \bmod 2, b \bmod 2)$ is surjective with precisely this kernel, so it gives the displayed quotient isomorphism. For comparison, the integer row operation $\text{row}_2\leftarrow \text{row}_2-2\text{row}_1$ changes $\phi$ to $\operatorname{diag}(2,6)$. Thus $A\cong\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/6\mathbb{Z}$, and tensoring the two summands with $\mathbb{Z}/4\mathbb{Z}$ gives the same two copies of $\mathbb{Z}/2\mathbb{Z}$.

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