Resolutions, Tor, and Ext

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3.1 Resolutions and Derived Functors

3.1.1 Projective and Injective Modules

The tensor-product construction in Lecture 2, Section 2.3, computes degree zero. To measure the exactness lost by tensoring, we extend a presentation to an exact sequence. The constructions in this section work for right modules over any ring; we continue with commutative R until Section 3.3.

Definition 3.1.1. Let $\mathcal{A}$ be an abelian category. An object $P \in \mathcal{A}$ is *projective* if for every epimorphism $q : E \to L$ in $\mathcal{A}$ and morphism $f : P \to L$, there exists a morphism $\tilde{f} : P \to E$ such that $q \circ \tilde{f} = f$. A right R-module is projective if it is a projective object of the abelian category of right R-modules.

Definition 3.1.2. Let $\mathcal{A}$ be an abelian category. An object $I \in \mathcal{A}$ is *injective* if for every monomorphism $j : A \to B$ in $\mathcal{A}$ and morphism $f : A \to I$, there exists a morphism $\tilde{f} : B \to I$ such that $\tilde{f} \circ j = f$. A right R-module is injective if it is an injective object of the abelian category of right R-modules.

Remark 3.1.3. The projectivity condition is equivalent to exactness of $\mathrm{Mor}_{\mathcal{A}}(P, -)$; injectivity is equivalent to exactness of $\mathrm{Mor}_{\mathcal{A}}(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$. In each case the lifting property gives the required surjection, while exactness at the other terms follows from the kernel or cokernel universal property. Free modules are projective: lift the images of their basis elements. A projective module is a direct summand of a free module because a free surjection onto it splits. Conversely, a direct summand of a free module has the lifting property.

Definition 3.1.4. An abelian category $\mathcal{A}$ has *enough projectives* if for every $M \in \mathcal{A}$ there exist a projective object $P \in \mathcal{A}$ and an epimorphism $P \to M$. It has *enough injectives* if for every $M \in \mathcal{A}$ there exist an injective object $I \in \mathcal{A}$ and a monomorphism $M \to I$.

Module categories have both properties. Free modules supply enough projectives; the following construction supplies enough injectives.

Proposition 3.1.5. Every right module over a unital ring embeds in an injective right module.

Proof. First, a divisible abelian group D is injective. Given a map from a subgroup A of an abelian group B, use Zorn's lemma to choose a maximal extension to a subgroup C. If $b \notin C$, the set of integers n with $nb \in C$ is an ideal. If it is zero, choose any image of b. Otherwise it is $d\mathbb{Z}$ for $d > 0$; choose $a \in D$ with $da = f(db)$ and set $f(c + nb) = f(c) + na$. The ideal description proves this is well defined. Both cases extend the map to $C + \mathbb{Z}b$, contradicting maximality. Hence $C = B$.

The group $\mathbb{Q}/\mathbb{Z}$ is divisible, and its characters separate points of any abelian group M: a nonzero element has a cyclic subgroup admitting a character nonzero on that element, which extends by injectivity. Therefore M embeds by evaluation into $D = (\mathbb{Q}/\mathbb{Z})^{\mathrm{Mor}_{\mathrm{Ab}}(M, \mathbb{Q}/\mathbb{Z})}$. This product is injective because each component of a map extends.

For a right R-module M, take such an embedding $j : M \hookrightarrow D$ of underlying abelian groups.

Put $E = \mathrm{Mor}_{\mathrm{Ab}}(R, D)$ and define $(\phi r)(s) = \phi(rs)$. The bijection

$$\mathrm{Mor}_{\mathrm{Mod}_R}(X, E) \cong \mathrm{Mor}_{\mathrm{Ab}}(X, D)$$

sends $f : X \to E$ to the map $X \to D$ whose value at a is $f(a)(1)$. Its inverse sends $g : X \to D$ to the map $X \to E$ whose value at a is the homomorphism $R \to D$ given by $s \mapsto g(as)$. Since forgetting the R-action is exact and D is injective, E is injective. The map $M \to E$ given by the formula $m \mapsto \phi_m$, where $\phi_m : R \to D$ is given by $\phi_m(s) = j(ms)$ is R-linear and injective, as evaluation at 1 recovers j. □

Example 3.1.6. The abelian group $\mathbb{Z}$ is free and therefore projective. It is not injective: the map $2\mathbb{Z} \to \mathbb{Z}$ sending 2 to 1 cannot extend to $\mathbb{Z}$. Both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective by divisibility. The nonzero group $\mathbb{Q}/\mathbb{Z}$ is not projective: projective abelian groups are summands of free abelian groups and hence torsion-free, whereas $\mathbb{Q}/\mathbb{Z}$ is torsion. Thus

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

is an injective resolution of $\mathbb{Z}$, with $\mathbb{Q}$ in cochain degree zero.

3.1.2 Projective Resolutions and Their Comparison

Definition 3.1.7. A projective resolution of M is an exact augmented sequence

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0 \tag{9}$$

with every P_i projective. The associated chain complex $P_\bullet$ has $P_i = 0$ for $i < 0$, differential $\partial_0 = 0$, $\mathrm{H}_0(P_\bullet) \cong M$, and $\mathrm{H}_i(P_\bullet) = 0$ for $i > 0$. The augmentation is not part of the unaugmented complex.

Every module has a free resolution. Choose a free surjection $P_0 \to M$; choose a free surjection $P_1 \to \ker \epsilon$; choose a free surjection $P_2 \to \ker \partial_1$; and continue. The successive kernels are called syzygies. For a completely general choice, use a free module on the underlying set of each kernel. The process may require infinite rank and may continue indefinitely.

Remark 3.1.8. A presentation supplies $\partial_1 : P_1 \to P_0$, but does not assert that ∂_1 is injective. To compute degree-one derived functors one generally also needs ∂_2, which records the relations among the original relations. To compute degree i, one needs the resolution through $P_{i+1} \to P_i \to P_{i-1}$.

Theorem 3.1.9. Let $P_\bullet \to M$ and $Q_\bullet \to L$ be projective resolutions. Every map $f : M \to L$ lifts to a chain map $\tilde{f} : P_\bullet \to Q_\bullet$. Any two lifts of the same f are chain homotopic. In particular, any two projective resolutions of M are chain homotopy equivalent by maps lifting id_M.

Proof. Lift $f\epsilon_P$ through $Q_0 \twoheadrightarrow L$ using projectivity of P_0. Having constructed the lift through degree $n-1$, the map $\tilde{f}_{n-1}\partial_n$ lands in the kernel of the next differential (or of the augmentation when $n = 1$). Exactness identifies this kernel with the image of Q_n. Projectivity of P_n gives $\tilde{f}_n$.

For two lifts α, β, construct $\gamma_n : P_n \to Q_{n+1}$ satisfying

$$\alpha_n - \beta_n = \partial^Q_{n+1}\gamma_n + \gamma_{n-1}\partial^P_n, \qquad \gamma_{-1} = 0.$$

In degree zero, $\alpha_0 - \beta_0$ is killed by the augmentation, so lift it through ∂^Q_1. Inductively, the chain-map identities and the formula already established in degree $n-1$ show that

$$\partial^Q_n(\alpha_n - \beta_n - \gamma_{n-1}\partial^P_n) = 0.$$

Exactness and projectivity give the required γ_n. Finally, lifts of id_M in both directions have composites homotopic to the appropriate identities, by the uniqueness statement. □

Theorem 3.1.10. Let R be a principal ideal domain. Every submodule of a free R-module is free, with no restriction on the rank.

Proof. Well-order a basis $(e_\alpha)_{\alpha<\kappa}$ of F and, for $N \subseteq F$, put $F_\beta = \bigoplus_{\alpha<\beta} Re_\alpha$ and $N_\beta = N \cap F_\beta$. The e_α-coordinate map identifies $N_{\alpha+1}/N_\alpha$ with an ideal $I_\alpha \subseteq R$. If $I_\alpha = (a_\alpha) \neq 0$, choose $x_\alpha \in N_{\alpha+1}$ with coordinate a_α. Subtracting multiples of x_α shows $N_{\alpha+1} = N_\alpha \oplus Rx_\alpha$: the sum is direct because $ra_\alpha = 0$ implies $r = 0$. If $I_\alpha = 0$, then $N_{\alpha+1} = N_\alpha$. At a limit ordinal λ, $N_\lambda = \bigcup_{\beta<\lambda} N_\beta$, since every vector has finite support. Transfinite induction therefore gives

$$N_\beta = \bigoplus_{\substack{\alpha<\beta \\ I_\alpha \neq 0}} Rx_\alpha.$$

Indeed the successor step is the displayed direct-sum decomposition; at a limit, every vector and every finite linear relation already occur at an earlier stage. Taking $\beta = \kappa$ gives the required basis of N. □

Corollary 3.1.11. Every module M over a principal ideal domain admits a free resolution $0 \to P_1 \to P_0 \to M \to 0$, allowing arbitrary ranks.

Proof. Take a free surjection $P_0 \to M$. Its kernel P_1 is free by Theorem 3.1.10. □

3.1.3 Left and Right Derived Functors

Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let all functors in this subsection be additive. The constructions apply to objects of $\mathcal{A}$; complexes of objects require a further construction. The construction by resolutions is developed further in Weibel [Wei94, Chapter 2, pp. 30–65].

Definition 3.1.12. Let $\mathcal{A}$ be an abelian category and $M \in \mathcal{A}$. An *injective resolution* of M is an exact sequence $0 \to M \to I^0 \to I^1 \to \cdots$ in which every I^q is injective. Its associated cochain complex $I^\bullet$ has $I^q = 0$ for $q < 0$; the augmentation $M \to I^0$ is not part of this complex.

Definition 3.1.13. If $\mathcal{A}$ has enough projectives and $F : \mathcal{A} \to \mathcal{B}$ is right exact, choose a projective resolution $P_\bullet \to M$ and set

$$\mathrm{L}_n F(M) = \mathrm{H}_n(F(P_\bullet)), \qquad n \geq 0.$$

If $\mathcal{A}$ has enough injectives and $G : \mathcal{A} \to \mathcal{B}$ is left exact, construct an exact sequence $0 \to M \to I^0 \to I^1 \to \cdots$ with each I^q injective, by successively embedding the cokernels. Set

$$\mathrm{R}^n G(M) = \mathrm{H}^n(G(I^\bullet)), \qquad n \geq 0.$$

For a morphism $f : M \to N$, choose a chain map $\tilde{f}$ between projective resolutions lifting f and define $\mathrm{L}_n F(f) = \mathrm{H}_n(F(\tilde{f}))$. Dually, choose a cochain map $\hat{f}$ between injective resolutions extending f and define $\mathrm{R}^n G(f) = \mathrm{H}^n(G(\hat{f}))$. These assignments define functors $\mathrm{L}_n F, \mathrm{R}^n G : \mathcal{A} \to \mathcal{B}$; their well-definedness is proved in Proposition 3.1.14. For a cochain complex, H^n means the kernel of the differential out of degree n modulo the image of the differential into degree n.

Proposition 3.1.14. These constructions give functors independent of the resolutions up to canonical natural isomorphism. There are natural isomorphisms $\mathrm{L}_0 F \cong F$ and $\mathrm{R}^0 G \cong G$. For $n > 0$, $\mathrm{L}_n F(P) = 0$ for projective P and $\mathrm{R}^n G(I) = 0$ for injective I.

Proof. The comparison theorem and its dual lift maps of objects to maps of resolutions uniquely up to chain or cochain homotopy. Additive functors preserve the homotopy equations,

so their induced homology maps are unique and respect identities and composition. Comparing identity lifts between two resolutions gives mutually inverse isomorphisms; uniqueness shows their compatibility with a third resolution. Right exactness identifies the cokernel of $F(P_1) \to F(P_0)$ with $F(M)$. Left exactness identifies the kernel of $G(I^0) \to G(I^1)$ with $G(M)$. Finally use the resolution concentrated in degree zero when the object is itself projective or injective, respectively. □

Lemma 3.1.15. Let $0 \to M' \to M \to M'' \to 0$ be exact in an abelian category. Given projective resolutions $P'_\bullet \to M'$ and $P''_\bullet \to M''$, there is a projective resolution $P_\bullet \to M$ fitting into a degreewise split exact sequence $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$. Dually, injective resolutions of M' and M'' extend to a degreewise split exact sequence $0 \to I'^\bullet \to I^\bullet \to I''^\bullet \to 0$ resolving the given sequence.

Proof. Start with $(K'_0, K_0, K''_0) = (M', M, M'')$. Inductively suppose $0 \to K'_n \xrightarrow{i_n} K_n \xrightarrow{\pi_n} K''_n \to 0$ is exact. The chosen end resolutions give epimorphisms $e'_n : P'_n \to K'_n$ and $e''_n : P''_n \to K''_n$ with kernels K'_{n+1} and K''_{n+1}. Projectivity gives a lift $t_n : P''_n \to K_n$ of e''_n. Set $P_n = P'_n \oplus P''_n$ and $e_n = (i_n e'_n, t_n)$. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_n & \hookrightarrow & P'_n \oplus P''_n & \twoheadrightarrow & P''_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle e'_n} & & \downarrow{\scriptstyle e_n} & & \downarrow{\scriptstyle e''_n} & & \\ 0 & \longrightarrow & K'_n & \xrightarrow[i_n]{} & K_n & \xrightarrow[\pi_n]{} & K''_n & \longrightarrow & 0 \end{array}$$

commutes. The map e_n is epi: if $ue_n = 0$, then $ui_n = 0$, so $u = v\pi_n$; now $ve''_n = ut_n = 0$ forces $v = 0$. Put $K_{n+1} = \ker e_n$. To verify the next exact sequence, write $j''_{n+1} : K''_{n+1} \hookrightarrow P''_n$ and factor $t_n j''_{n+1} = i_n h_n$. The kernel universal property identifies $K_{n+1} \to K''_{n+1}$ with the pullback of e'_n along $-h_n$. It is therefore epi by Lecture 1, Lemma 1.2.11, and its kernel is K'_{n+1}. Thus

$$0 \longrightarrow K'_{n+1} \longrightarrow K_{n+1} \longrightarrow K''_{n+1} \longrightarrow 0$$

is exact, completing the induction. If $j_n : K_n \hookrightarrow P_{n-1}$ is the kernel inclusion, define $d_n = j_n e_n$ for $n \geq 1$, with augmentation e_0. The equality $e_{n-1} j_n = 0$ gives $d^2 = 0$; monicity of j_n and epimorphicity of e_{n+1} give $\ker d_n = K_{n+1} = \operatorname{im} d_{n+1}$ inside P_n. The construction of the kernel maps makes inclusion and projection chain maps, and the objects P_n are projective.

For the dual, apply this proof in the opposite abelian category, where injectives are projectives. Explicitly, at stage n take $0 \to C'^n \xrightarrow{i^n} C^n \xrightarrow{\pi^n} C''^n \to 0$, starting with the given sequence. The end resolutions supply injections $a'^n : C'^n \to I'^n$ and $a''^n : C''^n \to I''^n$. Extend a'^n to $s^n : C^n \to I'^n$, put $I^n = I'^n \oplus I''^n$, and set

$$a^n = (s^n, a''^n \pi^n) : C^n \longrightarrow I'^n \oplus I''^n.$$

This is monic; the dual of the kernel argument proves that its cokernel fits into $0 \to C'^{n+1} \to C^{n+1} \to C''^{n+1} \to 0$. With $q^n : I^n \to C^{n+1}$ the cokernel map, the differentials $d^n = a^{n+1} q^n$ give the asserted exact resolutions and commuting maps. Thus the dual uses precisely the proved induction with arrows reversed.

These constructions also lift a morphism of short exact sequences. At each projective stage first lift the two end maps. The discrepancy on P''_n between the desired middle map and the chosen end lift lands in the target K'_n; lift it through the target $P'_n \twoheadrightarrow K'_n$ and add it as the off-diagonal component. This gives a commuting diagram and hence maps on the next kernels. The dual argument applies to injectives. The comparison theorem makes the induced derived maps independent of these choices, so the resulting long exact sequences are natural. □

Remark 3.1.16. Lemma 3.1.15 resolves a short exact sequence by a degreewise split short exact sequence of projective resolutions. Additivity preserves each splitting, so applying F

and taking homology gives

$$\cdots \to \mathrm{L}_1 F(M) \to \mathrm{L}_1 F(M'') \to F(M') \to F(M) \to F(M'') \to 0.$$

The dual construction with injectives gives

$$0 \to G(M') \to G(M) \to G(M'') \to \mathrm{R}^1 G(M') \to \mathrm{R}^1 G(M) \to \cdots.$$

In particular, an exact sequence $0 \to K \to P \to M \to 0$ with P projective gives $\mathrm{L}_{n+1} F(M) \cong \mathrm{L}_n F(K)$ for $n \geq 1$; an exact sequence $0 \to M \to I \to C \to 0$ with I injective gives $\mathrm{R}^{n+1} G(M) \cong \mathrm{R}^n G(C)$ for $n \geq 1$. These are the dimension-shifting isomorphisms.

3.2 Tor and the Failure of Exactness

3.2.1 Definition and Computation of Tor

Again let R be commutative. Tensoring the injection $2 : \mathbb{Z} \to \mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$ gives the zero map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, which is not injective. Tensor products preserve cokernels but can fail to preserve kernels.

Definition 3.2.1. Choose a projective resolution $P_\bullet \to M$. Define

$$\mathrm{Tor}_i^R(M, N) = \mathrm{H}_i(P_\bullet \otimes_R N) = \frac{\ker(\partial_i \otimes \mathrm{id}_N)}{\mathrm{im}(\partial_{i+1} \otimes \mathrm{id}_N)} \quad (i \geq 0),$$

where the differential out of degree zero is zero. These are R-modules. Right exactness gives

$$\mathrm{Tor}_0^R(M, N) \cong M \otimes_R N.$$

The comparison theorem makes these groups independent of the resolution up to canonical isomorphism: a chain homotopy remains a chain homotopy after tensoring. Lifting maps of M gives functoriality in the first variable; tensoring maps of N gives functoriality in the second. Thus Tor is covariant in both variables.

For a free resolution $P_i = \bigoplus_{\lambda \in \Lambda_i} R$, the computation is completely explicit:

$$\cdots \longrightarrow \bigoplus_{\lambda \in \Lambda_2} N \xrightarrow{(\partial_2)_N} \bigoplus_{\lambda \in \Lambda_1} N \xrightarrow{(\partial_1)_N} \bigoplus_{\lambda \in \Lambda_0} N \longrightarrow 0.$$

Use the same coefficient matrices as in the free resolution, now acting on direct sums of N; take kernels modulo images. A presentation by itself computes $\mathrm{Tor}_0^R(M, N)$, while $\mathrm{Tor}_1^R(M, N)$ generally requires the next map.

Example 3.2.2. Suppose multiplication by $a \in R$ is injective on R. Then $0 \to R \xrightarrow{a} R \to R/(a) \to 0$ is a free resolution. Writing $N[a] = \{n \in N : na = 0\}$, we obtain

$$\mathrm{Tor}_0^R(R/(a), N) = N/Na, \qquad \mathrm{Tor}_1^R(R/(a), N) = N[a], \qquad \mathrm{Tor}_i^R(R/(a), N) = 0 \quad (i \geq 2).$$

In particular, for $a, b > 0$, $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z}) \cong \mathbb{Z}/\gcd(a, b)\mathbb{Z}$. The kernel description is natural; an identification with a standard cyclic group involves a choice of generator.

Example 3.2.3. For the matrix in Lecture 2, Example 2.3.8, $\det \phi = 12 \neq 0$, so $\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ is injective. It therefore gives a length-one free resolution of M. Modulo four, $\phi : (\mathbb{Z}/4\mathbb{Z})^2 \to (\mathbb{Z}/4\mathbb{Z})^2$ acts by $(a, b) \mapsto (2a, 2b)$, whose kernel is $\{0, 2\}^2$. Thus

$$\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/4\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2, \qquad \mathrm{Tor}_i^{\mathbb{Z}}(M, \mathbb{Z}/4\mathbb{Z}) = 0 \quad (i \geq 2).$$

Indeed, $\phi(a,b) = 0$ over $\mathbb{Z}$ implies $2a = 0$, hence $a = 0$, and then $6b = 0$, hence $b = 0$. The tensor complex is $0 \to (\mathbb{Z}/4\mathbb{Z})^2 \xrightarrow{(2a,2b)} (\mathbb{Z}/4\mathbb{Z})^2 \to 0$, in degrees one and zero. A degree-one cycle has each coordinate either zero or two. There is no degree-two term and hence no nonzero degree-one boundary. The map $(\mathbb{Z}/2\mathbb{Z})^2 \to (\mathbb{Z}/4\mathbb{Z})^2$ given by $(\bar{a},\bar{b}) \mapsto (2a,2b)$ identifies $(\mathbb{Z}/2\mathbb{Z})^2$ with this kernel. In degree zero the cokernel was computed in Lecture 2, Example 2.3.8.

Example 3.2.4. Let $R = k[\varepsilon]/(\varepsilon^2)$ for a field k, and let $K = R/(\varepsilon)$. Since the kernel and image of multiplication by ε are both (ε),

$$\cdots \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R \longrightarrow K \longrightarrow 0$$

is a free resolution. To check this directly, write an element of R uniquely as $a + b\varepsilon$. Multiplication by ε sends it to $a\varepsilon$, so its kernel and image are both $k\varepsilon$. The augmentation has the same kernel. Since $K\varepsilon = 0$, tensoring gives one copy of k in each nonnegative degree and zero differential. Every vector is a cycle and the only boundary is zero, so $\operatorname{Tor}_i^R(K,K) \cong k$ for every $i \geq 0$. The non-zero-divisor hypothesis in Example 3.2.2 cannot be dropped.

3.2.2 Flatness and Long Exact Sequences

Definition 3.2.5. A module F is flat if $- \otimes_R F$ is exact. Since tensoring is right exact, this means that it preserves injections. Free modules are flat because tensoring with a free module is a direct sum of copies of the original sequence; direct sums of exact sequences of modules are exact. Direct summands of flat modules are flat, so projective modules are flat.

Theorem 3.2.6. If $P_\bullet \to M$ and $Q_\bullet \to N$ are projective resolutions, there are natural isomorphisms

$$\mathrm{H}_i(P_\bullet \otimes_R N) \cong \mathrm{H}_i(M \otimes_R Q_\bullet).$$

Consequently $\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_i^R(N,M)$ naturally.

Proof. Form the first-quadrant double complex $P_p \otimes_R Q_q$ and its total complex

$$T_n = \bigoplus_{p+q=n} P_p \otimes_R Q_q, \qquad \partial(\alpha \otimes \beta) = \partial_P \alpha \otimes \beta + (-1)^p \alpha \otimes \partial_Q \beta.$$

Write $\partial_h = \partial_P \otimes \mathrm{id}$ and $\partial_v = \mathrm{id} \otimes \partial_Q$, so the total differential on column p is $\partial_h + (-1)^p \partial_v$. The sign makes the two mixed terms in ∂^2 cancel. Since P_p is flat, each augmented column is exact, with degree-zero homology $P_p \otimes_R N$. Similarly, each augmented row is exact, with degree-zero homology $M \otimes_R Q_q$.

Write $\eta : Q_0 \to N$ for the resolution augmentation and $u : T \to P_\bullet \otimes_R N$ for the induced total-complex augmentation. In degree n it is induced by $\mathrm{id}_{P_n} \otimes \eta$ on the summand with $q = 0$ and is zero on the other summands. It is a degreewise surjective chain map. Its kernel is the total complex of the double complex obtained by replacing the bottom term of column p by

$$K_{p,0} = \ker(P_p \otimes_R Q_0 \longrightarrow P_p \otimes_R N)$$

and retaining $K_{p,q} = P_p \otimes_R Q_q$ for $q > 0$. Each resulting column is exact, including at degree zero; the horizontal differential preserves these kernels.

Let ζ be a total cycle of degree n in this kernel, and choose its largest nonzero column index p. Put $q = n - p$. The cycle equation in bidegree $(p, q-1)$ says that the vertical differential of $\zeta_{p,q}$ is zero, since the component in column $p+1$ vanishes. When $q = 0$, the vertical differential out of the bottom term is zero by definition. Column exactness gives $\omega \in K_{p,q+1}$ with

$$\partial_v \omega = (-1)^p \zeta_{p,q}.$$

The total boundary of ω has component $\zeta_{p,q}$ in column p and its only other possible component in column $p-1$. Subtract it from ζ. Repeating strictly decreases the largest nonzero column index until the cycle is zero. Thus every cycle in $\ker u$ bounds. The long exact sequence of $0 \to \ker u \to T \to P_\bullet \otimes_R N \to 0$ proves that u is a quasi-isomorphism.

For $v : T \to M \otimes_R Q_\bullet$, replace the leftmost term of each row by the kernel of its augmentation. These rows are exact. Eliminate a total cycle starting with its largest row index q, choosing a horizontal primitive. Its total boundary has the prescribed component in that row and its only other possible component in row $q-1$, with the sign already specified in the total differential. The same finite induction proves that $\ker v$ is acyclic. Both augmentations commute with comparison maps of resolutions, so their homology isomorphisms are natural. Taking homology gives the first assertion, and interchanging the tensor factors gives symmetry. □

Theorem 3.2.7. For a short exact sequence $0 \to N' \to N \to N'' \to 0$, there is a natural long exact sequence. For $i \geq 1$, five consecutive terms are

$$\operatorname{Tor}_i^R(M,N') \to \operatorname{Tor}_i^R(M,N) \to \operatorname{Tor}_i^R(M,N'') \xrightarrow{\delta} \operatorname{Tor}_{i-1}^R(M,N') \to \operatorname{Tor}_{i-1}^R(M,N).$$

Using the identification $\operatorname{Tor}_0^R(M,N) \cong M \otimes_R N$, its final five terms are

$$\operatorname{Tor}_1^R(M,N'') \xrightarrow{\delta} M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0.$$

There is also a long exact sequence in the first variable. A module F is flat if and only if $\operatorname{Tor}_1^R(M,F) = 0$ for every M; in that case all $\operatorname{Tor}_i^R(M,F)$ vanish for $i > 0$.

Proof. Each term of a projective resolution of M is flat, so tensoring the short exact sequence gives a degreewise short exact sequence of chain complexes. Its homology long exact sequence is the displayed one. Theorem 3.2.6 gives the sequence in the first variable.

If F is flat, tensoring the augmented resolution of M preserves exactness, so all positive homology vanishes. Conversely, apply the long exact sequence in the first variable to $0 \to A \to B \to C \to 0$. Vanishing of $\operatorname{Tor}_1^R(C,F)$ implies that $A \otimes_R F \to B \otimes_R F$ is injective. This is flatness. □

Remark 3.2.8. For a particular short exact sequence, the kernel of $M \otimes_R N' \to M \otimes_R N$ is the image of the connecting map from $\operatorname{Tor}_1^R(M,N'')$. That connecting map need not be injective. Thus the kernel is a quotient of this Tor group, not generally the whole group.

3.3 Cochain Complexes and Ext Groups

3.3.1 Cochain Complexes and Morphism Functors

From now on, R is an arbitrary unital ring and all modules are right R-modules. A cochain complex uses an increasing differential. No commutativity is needed for its definition or for the usual exact-sequence arguments in R-modules.

Definition 3.3.1. A cochain complex $A^\bullet$ consists of maps $\partial_A^n : A^n \to A^{n+1}$ with $\partial_A^{n+1}\partial_A^n = 0$. Its cocycles, coboundaries, and cohomology are

$$Z^n(A) = \ker \partial_A^n, \qquad B^n(A) = \operatorname{im} \partial_A^{n-1}, \qquad \mathrm{H}^n(A) = Z^n(A)/B^n(A).$$

A cochain map $f : A \to B$ satisfies $\partial_B^n f^n = f^{n+1}\partial_A^n$. It induces a map on cohomology. It is a quasi-isomorphism when those maps are isomorphisms for all n.

Reindexing a chain complex by $A^n = A_{-n}$ and $\partial^n = \partial_{-n}$ gives a cochain complex, with $\mathrm{H}^n(A^\bullet) = \mathrm{H}_{-n}(A_\bullet)$. By contrast, applying the contravariant morphism functor to a chain complex reverses its arrows and gives cochains in nonnegative degrees; that is the construction used for Ext.

Definition 3.3.2. A cochain homotopy from g to f is a family $h^n : A^n \to B^{n-1}$ with

$$f^n - g^n = \partial_B^{n-1} h^n + h^{n+1} \partial_A^n.$$

On cocycles the difference is a coboundary, so homotopic maps induce the same cohomology maps. The shift is $A[r]^n = A^{n+r}$ with differential $(-1)^r \partial_A^{n+r}$. In particular, $\mathrm{H}^n(A[r]) = \mathrm{H}^{n+r}(A)$.

For a cochain map $f : A \to B$, its cone is

$$\mathrm{Cone}(f)^n = B^n \oplus A^{n+1}, \qquad \partial^n = \begin{pmatrix} \partial_B^n & f^{n+1} \\ 0 & -\partial_A^{n+1} \end{pmatrix}.$$

The chain-map equation makes this differential square to zero. Inclusion of the first summand and projection onto the second give $0 \to B \to \mathrm{Cone}(f) \to A[1] \to 0$.

Proposition 3.3.3. A degreewise short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\rho} C \to 0$ of cochain complexes gives a long exact sequence. Five consecutive terms are

$$\mathrm{H}^n(A) \to \mathrm{H}^n(B) \to \mathrm{H}^n(C) \xrightarrow{\delta} \mathrm{H}^{n+1}(A) \to \mathrm{H}^{n+1}(B).$$

For a cocycle $\zeta \in C^n$, choose $\beta \in B^n$ with $\rho\beta = \zeta$ and write $\partial_B \beta = \iota\alpha$. Then $\delta[\zeta] = [\alpha]$.

Proof. The equation $\rho \partial_B \beta = \partial_C \zeta = 0$ gives α, and $\iota \partial_A \alpha = \partial_B^2 \beta = 0$ makes it a cocycle. Changing β by $\iota\alpha'$ changes α by $\partial_A \alpha'$; changing ζ by a coboundary and lifting its primitive does not change the resulting cohomology class. The usual kernel–image chase, or the snake lemma on kernels and cokernels of consecutive differentials, gives exactness. Equivalently, reindex all three complexes as chain complexes and apply the homology long exact sequence; its connecting map is exactly this lift-and-differentiate formula. □

Definition 3.3.4. For a chain complex $P_\bullet$ in nonnegative degrees and a module N, put

$$\mathrm{Mor}_{\mathrm{Mod}_R}(P_\bullet, N)^n = \mathrm{Mor}_{\mathrm{Mod}_R}(P_n, N), \qquad \partial^n(\phi) = \phi \circ \partial_{n+1}.$$

Thus the arrows run as follows:

$$0 \to \mathrm{Mor}_{\mathrm{Mod}_R}(P_0, N) \xrightarrow{\partial^0} \mathrm{Mor}_{\mathrm{Mod}_R}(P_1, N) \xrightarrow{\partial^1} \mathrm{Mor}_{\mathrm{Mod}_R}(P_2, N) \to \cdots.$$

The relation $\partial^{n+1}\partial^n = 0$ follows from $\partial_{n+1}\partial_{n+2} = 0$. For general R, this is a cochain complex of abelian groups, canonically of modules over the center $Z(R)$. It is not generally a complex of right R-modules: multiplying the values of an R-linear map on the right by a noncentral element need not give an R-linear map.

Remark 3.3.5. Write $\mathrm{Ch}^\bullet(\mathrm{Mod}_R)$ for cochain complexes of right R-modules. For cochain complexes A, B, define the mapping complex of abelian groups

$$\underline{\mathrm{Mor}}_{\mathrm{Ch}^\bullet(\mathrm{Mod}_R)}(A, B)^n = \prod_p \mathrm{Mor}_{\mathrm{Mod}_R}(A^p, B^{p+n}), \qquad (\partial f)^p = \partial_B^{p+n} f^p - (-1)^n f^{p+1} \partial_A^p.$$

Its degree-zero cocycles are cochain maps, and its degree-zero coboundaries are nullhomotopic maps. If $A^{-n} = P_n$ and $B = N$ is concentrated in degree zero, this convention gives $\partial\phi = (-1)^{n+1} \phi \partial_{n+1}$. Multiplication in degree n by $(-1)^{n(n+1)/2}$ identifies the unsigned complex in Definition 3.3.4 with this signed convention. We use the unsigned complex for the Ext computations below.

3.3.2 Ext Groups, Variance, and Computation

The ring R remains arbitrary. Projective resolutions of right R-modules exist by Section 3.1.

Definition 3.3.6. For right modules M, N, choose a projective resolution $P_\bullet \to M$ and define

$$\operatorname{Ext}_R^n(M,N) = \mathrm{H}^n(\operatorname{Mor}_{\mathrm{Mod}_R}(P_\bullet, N)) = \frac{\ker(\partial^n)}{\operatorname{im}(\partial^{n-1})} \quad (n \geq 0),$$

where the image in degree zero is zero. These are abelian groups with a natural $Z(R)$-module structure; if R is an algebra over a field k, they are k-vector spaces. When R is commutative, they are naturally R-modules.

The comparison theorem again proves independence of resolutions. More explicitly, if $f, g : P_\bullet \to Q_\bullet$ satisfy $f - g = \partial_Q h + h\partial_P$, precomposition gives cochain-homotopic maps: on $\phi : Q_n \to N$ use the homomorphism $\operatorname{Mor}_{\mathrm{Mod}_R}(Q_n, N) \to \operatorname{Mor}_{\mathrm{Mod}_R}(P_{n-1}, N)$ given by $\phi \mapsto \phi h_{n-1}$. Thus Ext is contravariant in M and covariant in N. In degree zero, a map $P_0 \to N$ lies in $\ker \partial^0$ exactly when it kills $\operatorname{im} \partial_1$, so it factors through M and

$$\operatorname{Ext}_R^0(M,N) \cong \operatorname{Mor}_{\mathrm{Mod}_R}(M,N).$$

For an arbitrary free resolution $P_n = \bigoplus_{\lambda \in \Lambda_n} R$, note the essential change from tensor products:

$$\operatorname{Mor}_{\mathrm{Mod}_R}\Big(\bigoplus_{\lambda \in \Lambda_n} R, N\Big) \cong \prod_{\lambda \in \Lambda_n} N.$$

Maps out of a free direct sum may have arbitrary values on its basis, so this is a product, not a direct sum. If $\partial_{n+1}(e_{\lambda'}) = \sum_{\lambda \in \Lambda_n} e_\lambda r_{\lambda\lambda'}$ for $\lambda' \in \Lambda_{n+1}$, with coefficients on the right, then

$$\partial^n((n_\lambda)_{\lambda \in \Lambda_n}) = \left(\sum_{\lambda \in \Lambda_n} n_\lambda r_{\lambda\lambda'}\right)_{\lambda' \in \Lambda_{n+1}}.$$

Every sum is finite because each column of the differential has finite support. Compute Ext by taking the kernel of this map modulo the image of the preceding map. Over a commutative ring with finite free terms, these are the transposed differential matrices acting on finite powers of N. For a noncommutative ring, the displayed order of the coefficients specifies the maps unambiguously.

Proposition 3.3.7. For $0 \to N' \to N \to N'' \to 0$, there is a natural long exact sequence. For $n \geq 0$, five consecutive terms are

$$\operatorname{Ext}_R^n(M,N') \to \operatorname{Ext}_R^n(M,N) \to \operatorname{Ext}_R^n(M,N'') \xrightarrow{\delta} \operatorname{Ext}_R^{n+1}(M,N') \to \operatorname{Ext}_R^{n+1}(M,N).$$

Recall that $\operatorname{Ext}_R^0(M,N) \cong \operatorname{Mor}_{\mathrm{Mod}_R}(M,N)$. The initial map $\operatorname{Mor}_{\mathrm{Mod}_R}(M,N') \to \operatorname{Mor}_{\mathrm{Mod}_R}(M,N)$ is injective. If P is projective, $\operatorname{Ext}_R^n(P,N) = 0$ for $n > 0$. Conversely, if $\operatorname{Ext}_R^1(M,N) = 0$ for every N, then M is projective.

Proof. Apply $\operatorname{Mor}_{\mathrm{Mod}_R}(P_i, -)$ to the short exact sequence. Each resulting sequence is exact by projectivity, so Proposition 3.3.3, applied in abelian groups, gives the displayed sequence. A projective P has a resolution concentrated in degree zero, proving its higher vanishing. For the converse, take a free surjection $F \to M$ with kernel K. The exact sequence makes $\operatorname{Mor}_{\mathrm{Mod}_R}(M,F) \to \operatorname{Mor}_{\mathrm{Mod}_R}(M,M)$ surjective because $\operatorname{Ext}_R^1(M,K) = 0$. A preimage of id_M splits $F \to M$, so M is projective. □

There is also a long exact sequence in the first, contravariant variable. For $0 \to M' \to M \to M'' \to 0$ and $n \geq 0$, five consecutive terms are

$$\operatorname{Ext}^n_R(M'', N) \to \operatorname{Ext}^n_R(M, N) \to \operatorname{Ext}^n_R(M', N) \xrightarrow{\delta} \operatorname{Ext}^{n+1}_R(M'', N) \to \operatorname{Ext}^{n+1}_R(M, N).$$

Under the same degree-zero identification, the initial map $\operatorname{Mor}_{\operatorname{Mod}_R}(M'', N) \to \operatorname{Mor}_{\operatorname{Mod}_R}(M, N)$ is injective. Apply Lemma 3.1.15 to obtain compatible resolutions with $P_i = P'_i \oplus P''_i$. Applying $\operatorname{Mor}_{\operatorname{Mod}_R}(-, N)$ gives a degreewise split short exact sequence of cochain complexes in the reverse order. Proposition 3.3.3 gives the displayed sequence, and the compatible comparison maps in the horseshoe proof give its naturality.

3.3.3 Extensions as Degree-One Classes

Theorem 3.3.8. The group $\operatorname{Ext}^1_R(M, N)$ classifies short exact sequences of right R-modules

$$0 \longrightarrow N \xrightarrow{\iota} E \xrightarrow{\rho} M \longrightarrow 0 \tag{10}$$

up to isomorphisms of sequences that are the identity on N and M. The zero class consists precisely of the split extensions.

Proof. Choose $0 \to K \xrightarrow{\jmath} P_0 \to M \to 0$ with P_0 projective. The definition of Ext from a resolution identifies

$$\operatorname{Ext}^1_R(M, N) \cong \operatorname{coker}(\operatorname{Mor}_{\operatorname{Mod}_R}(P_0, N) \longrightarrow \operatorname{Mor}_{\operatorname{Mod}_R}(K, N)). \tag{11}$$

Indeed, a map $P_1 \to N$ is a cocycle exactly when it kills $\operatorname{im}(\partial_2)$, so it factors through $P_1/\operatorname{im}(\partial_2) \cong K$.

Given $\alpha : K \to N$, form the quotient

$$E_\alpha = (N \oplus P_0)/\operatorname{im}(-\alpha, \jmath).$$

The construction fits into the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & K & = & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (-\alpha,\jmath)} & & \downarrow{\scriptstyle \jmath} & & \\
0 & \longrightarrow & N & \xrightarrow{\iota} & N \oplus P_0 & \xrightarrow{\mathrm{pr}_2} & P_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \epsilon} & & \\
0 & \longrightarrow & N & \xrightarrow{\iota_\alpha} & E_\alpha & \xrightarrow{\pi_\alpha} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

The middle column exhibits $E_\alpha = \operatorname{coker}(K \xrightarrow{(-\alpha,\jmath)} N \oplus P_0)$. Here $\iota : N \to N \oplus P_0$ satisfies $\iota(a) = (a, 0)$; $q(a, p) = [(a, p)]$, $\iota_\alpha(a) = [(a, 0)]$, and $\pi_\alpha([(a, p)]) = \epsilon(p)$. Thus the trivial extension of P_0 by N maps surjectively onto the desired extension of M by N. The map ι_α is injective because $\jmath$ is injective. If $\epsilon(p) = 0$, write $p = \jmath(k)$; then

$$[(a, p)] = [(a + \alpha(k), 0)],$$

which proves exactness at E_α.

Conversely, given an extension $0 \to N \xrightarrow{\iota} E \to M \to 0$, projectivity gives a lift $s : P_0 \to E$ of ϵ. There is a unique $\alpha : K \to N$ with $s\jmath = \iota\alpha$. The surjection

$$N \oplus P_0 \longrightarrow E, \qquad (a, p) \longmapsto \iota(a) + s(p)$$

has kernel $\operatorname{im}(-\alpha, \jmath)$, so it identifies the given extension with the bottom row of the diagram.

Changing the lift replaces α by $\alpha + \beta\jmath$ for some $\beta : P_0 \to N$. Conversely, if $\alpha' = \alpha + \beta\jmath$, the map $N \oplus P_0 \to N \oplus P_0$ given by $(a, p) \mapsto (a - \beta(p), p)$ descends to an equivalence $E_\alpha \cong E_{\alpha'}$. An equivalence of extensions also compares their lifts in exactly this way, so the equivalence classes are precisely the displayed cokernel. Finally, $E_0 \cong N \oplus M$, and a split extension admits a lift with $\alpha = 0$. Hence the zero class corresponds precisely to split extensions. □

The addition transported by this classification is the Baer sum: take the direct sum of two extensions, pull back along the diagonal $M \to M \oplus M$, and push out along addition $N \oplus N \to N$. In the description above, these operations replace the two maps $\alpha, \beta : K \to N$ by $\alpha + \beta$.

3.4 Computations of Ext

3.4.1 An Explicit Noncommutative Ring

Example 3.4.1. Let k be a field and

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in k \right\}, \qquad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This ring is noncommutative: $e_1 u = u$ whereas $u e_1 = 0$. Define right modules $S_1 = S_2 = k$ as vector spaces, with actions

$$\mu \cdot \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \mu a \quad \text{on } S_1, \qquad \mu \cdot \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \mu c \quad \text{on } S_2.$$

The right ideals $P_1 = e_1 R$ and $P_2 = e_2 R$ are projective because $R = e_1 R \oplus e_2 R$. They have bases e_1, u and e_2, respectively. There is a projective resolution

$$0 \longrightarrow P_2 \xrightarrow{\partial} P_1 \xrightarrow{\rho} S_1 \longrightarrow 0, \qquad \partial(\zeta) = u\zeta, \qquad \rho(a e_1 + b u) = a. \tag{12}$$

Left multiplication by u is a map of right modules. It sends e_2 to u, so it is injective with image $ku = \ker \rho$.

For any right R-module N, evaluation at e_i gives

$$\operatorname{Mor}_{\operatorname{Mod}_R}(e_i R, N) \cong N e_i.$$

Indeed, $f(e_i) = f(e_i)e_i$, and conversely $n \in N e_i$ determines $f_n(e_i r) = nr$. This formula is well defined: if $e_i r = e_i r'$, then $n(r - r') = n e_i (r - r') = 0$. Under these identifications, precomposition with ∂ is

$$N e_1 \longrightarrow N e_2, \qquad n \longmapsto nu,$$

because $\partial(e_2) = u = e_1 u$. Thus the entire Ext computation is the two-term cochain complex

$$N e_1 \xrightarrow{\cdot u} N e_2 \qquad \text{in degrees } 0, 1.$$

Consequently

$$\begin{aligned}\mathrm{Mor}_{\mathrm{Mod}_R}(S_1, N) &\cong \ker(\cdot u : Ne_1 \to Ne_2),\\ \mathrm{Ext}^1_R(S_1, N) &\cong Ne_2/(Ne_1)u,\\ \mathrm{Ext}^n_R(S_1, N) &= 0 \qquad (n \geq 2).\end{aligned}$$

For $N = S_2$, the actions give $S_2e_1 = 0$ and $S_2e_2 = k$. Thus the cochain complex is $0 \to 0 \to k \to 0$, in degrees zero and one. Its degree-zero cohomology is zero, and its degree-one cohomology is $k/0 = k$. Hence

$$\mathrm{Ext}^1_R(S_1, S_2) \cong k.$$

For $N = S_1$, the complex instead is $0 \to k \to 0 \to 0$, so $\mathrm{Mor}_{\mathrm{Mod}_R}(S_1, S_1) = k$ and $\mathrm{Ext}^1_R(S_1, S_1) = 0$. For $N = P_1$, we have $P_1e_1 = ke_1$ and $P_1e_2 = ku$; the differential $P_1e_1 \to P_1e_2$ given by $ae_1 \mapsto au$ is an isomorphism, so both cohomology groups vanish. By contrast, $S_2 \cong P_2$ is projective, so $\mathrm{Ext}^1_R(S_2, S_1) = 0$. Unlike Tor over a commutative ring, Ext is not symmetric in its variables.

The resolution (12) itself, after identifying P_2 with S_2, is a nonsplit extension of S_1 by S_2. If it had a section, the image v of $1 \in S_1$ would satisfy $\rho(v) = 1$, so $v = e_1 + bu$ for some $b \in k$. But u acts trivially on S_1, while $vu = u \neq 0$, contradicting linearity of the section.

More explicitly, the class $\lambda \in k$ corresponds to $E_\lambda = k\xi \oplus k\eta$ with

$$\xi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = c\xi, \qquad \eta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a\eta + b\lambda\xi.$$

Define $\iota : S_2 \to E_\lambda$ by $\iota(1) = \xi$ and $\rho : E_\lambda \to S_1$ by $\rho(\xi) = 0$, $\rho(\eta) = 1$. The sequence $0 \to S_2 \xrightarrow{\iota} E_\lambda \xrightarrow{\rho} S_1 \to 0$ has class λ under (11): the lift $P_1 \to E_\lambda$ determined by $e_1 \mapsto \eta$ sends $\partial(e_2) = u$ to $\eta u = \lambda\xi$. It splits exactly when $\lambda = 0$; $E_1 \cong P_1$ via $\xi \mapsto u$ and $\eta \mapsto e_1$. Nonzero values can give isomorphic middle modules while representing different extension classes, because equivalence of extensions must fix both end modules.

3.4.2 Kernels and Cokernels for Cyclic Groups

Example 3.4.2. The free resolution $0 \to \mathbb{Z} \xrightarrow{12} \mathbb{Z} \to \mathbb{Z}/12\mathbb{Z} \to 0$ is exact because multiplication by 12 is injective and its cokernel is $\mathbb{Z}/12\mathbb{Z}$. Set $N = \mathbb{Z}/8\mathbb{Z}$. Tensoring gives

$$0 \longrightarrow N \xrightarrow{4} N \longrightarrow 0 \quad \text{in chain degrees } 1, 0.$$

The image is $\{0, 4\}$, and the kernel is $\{0, 2, 4, 6\}$. The quotient map $N \to \mathbb{Z}/4\mathbb{Z}$ has kernel $\{0, 4\}$; hence $\mathbb{Z}/12\mathbb{Z} \otimes_{\mathbb{Z}} N \cong \mathbb{Z}/4\mathbb{Z}$. The map $\mathbb{Z}/4\mathbb{Z} \to N$, $[t]_4 \mapsto [2t]_8$, is injective with image $\{0, 2, 4, 6\}$. There is no incoming degree-two boundary, so it identifies $\mathrm{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/12\mathbb{Z}, N)$ with $\mathbb{Z}/4\mathbb{Z}$. Higher Tor vanishes.

Applying $\mathrm{Mor}_{\mathrm{Ab}}(-, N)$ reverses the arrows and gives

$$0 \longrightarrow N \xrightarrow{4} N \longrightarrow 0 \quad \text{in cochain degrees } 0, 1.$$

Thus $\mathrm{Mor}_{\mathrm{Ab}}(\mathbb{Z}/12\mathbb{Z}, N) = \{0, 2, 4, 6\}$, while $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/12\mathbb{Z}, N) = N/\{0, 4\} \cong \mathbb{Z}/4\mathbb{Z}$. Higher Ext vanishes. The same map is being used, but homological and cohomological grading put its kernel and cokernel in different degrees.

For general $a, b > 0$, put $g = \gcd(a, b)$, $a = ga'$, and $b = gb'$. The image of multiplication by a on $\mathbb{Z}/b\mathbb{Z}$ equals the subgroup generated by g, by Bézout's identity. Its cokernel is therefore $\mathbb{Z}/g\mathbb{Z}$. For the kernel, $b \mid ac$ is equivalent to $b' \mid c$, since $\gcd(a', b') = 1$. Its elements are the g multiples

of b' modulo b. The map $\mathbb{Z}/g\mathbb{Z} \to \mathbb{Z}/b\mathbb{Z}$ given by $[t]_g \mapsto [b't]_b$ gives its cyclic-group identification. This computes tensor, Tor, morphism groups, and Ext in the preceding manner.

Example 3.4.3. For $M = \operatorname{coker}\left(\begin{smallmatrix} 2 & 0 \\ 4 & 6 \end{smallmatrix}\right)$ and $N = \mathbb{Z}/4\mathbb{Z}$, a homomorphism $\mathbb{Z}^2 \to N$ is specified by a row (a, b). Precomposition with ϕ sends it to $(2a+4b, 6b)$. Equivalently, on column coordinates the cochain differential is ϕ^{T}. Modulo four this is the map $N^2 \to N^2$ given by $(a, b) \mapsto (2a, 2b)$. Its kernel is $\{0, 2\}^2$, and its image is also $\{0, 2\}^2$ in the next copy of N^2. Consequently

$$\operatorname{Mor}_{\mathrm{Ab}}(M, N) \cong (\mathbb{Z}/2\mathbb{Z})^2, \qquad \operatorname{Ext}^1_{\mathbb{Z}}(M, N) \cong (\mathbb{Z}/2\mathbb{Z})^2,$$

and all higher groups vanish. The map defining Ext is the transposed map, even though in this particular example reduction modulo four makes it coincide with the map used for Tor.

3.4.3 Computing Ext with Injective Resolutions

For a fixed right R-module M, the functor $\operatorname{Mor}_{\mathrm{Mod}_R}(M, -)$ is left exact. Its right derived functors are the Ext groups defined above.

Proposition 3.4.4. If $P_\bullet \to M$ is a projective resolution and $0 \to N \to I^\bullet$ is an injective resolution, there are natural isomorphisms

$$\mathrm{H}^n \operatorname{Mor}_{\mathrm{Mod}_R}(P_\bullet, N) \cong \mathrm{H}^n \operatorname{Mor}_{\mathrm{Mod}_R}(M, I^\bullet) = \mathrm{R}^n \operatorname{Mor}_{\mathrm{Mod}_R}(M, -)(N).$$

Proof. Use the first-quadrant double complex $C^{p,q} = \operatorname{Mor}_{\mathrm{Mod}_R}(P_p, I^q)$. Its horizontal differential ∂_h is precomposition with ∂_{p+1}, and its vertical differential ∂_v is postcomposition with the differential of I. These maps commute; on the total complex take $\partial = \partial_h + (-1)^p \partial_v$. Thus $\partial^2 = 0$. The total degree-n group is $\bigoplus_{p+q=n} C^{p,q}$, a finite sum.

For fixed p, exactness of $\operatorname{Mor}_{\mathrm{Mod}_R}(P_p, -)$ makes the augmented column $0 \to \operatorname{Mor}_{\mathrm{Mod}_R}(P_p, N) \to C^{p,0} \to C^{p,1} \to \cdots$ exact. For fixed q, injectivity of I^q makes the augmented row $0 \to \operatorname{Mor}_{\mathrm{Mod}_R}(M, I^q) \to C^{0,q} \to C^{1,q} \to \cdots$ exact. Both augmentations are injective cochain maps into the total complex: the first occupies the row $q = 0$, the second the column $p = 0$. Consider the quotient by the first. It is the total complex of the double complex with

$$D^{p,0} = C^{p,0} / \operatorname{Mor}_{\mathrm{Mod}_R}(P_p, N), \qquad D^{p,q} = C^{p,q} \quad (q > 0).$$

Its columns are exact, including injectivity of the differential out of degree zero. The horizontal differential descends to these quotients.

For a total cocycle ζ of degree n in this quotient, choose its largest nonzero vertical degree q, and put $p = n - q$. Its component in bidegree $(p, q+1)$ under the total differential is $(-1)^p \partial_v \zeta^{p,q}$: the other possible contribution has vertical degree $q + 1$ before differentiation and vanishes. Hence $\partial_v \zeta^{p,q} = 0$. If $q > 0$, choose $\omega \in D^{p,q-1}$ with

$$\partial_v \omega = (-1)^p \zeta^{p,q}.$$

Subtracting $\partial\omega$ removes that component and introduces a possible component only in vertical degree $q - 1$. Repeat. At $q = 0$, injectivity of the bottom vertical differential makes the remaining component zero. Thus the quotient total complex is acyclic. The cohomology long exact sequence proves that the first augmentation is a quasi-isomorphism.

For the second augmentation take the quotient of $C^{0,q}$ by $\operatorname{Mor}_{\mathrm{Mod}_R}(M, I^q)$ instead. Its rows are exact. Starting with the largest horizontal degree p, choose a horizontal primitive in degree $p - 1$; subtracting its total coboundary introduces a component only in horizontal degree $p - 1$. At $p = 0$ the bottom horizontal differential is injective. This proves the other quotient acyclic and the other augmentation a quasi-isomorphism. The two augmentation

diagrams commute with comparison maps of resolutions. Their homology isomorphisms, and therefore the resulting comparison, are natural. □

Example 3.4.5. To compute $\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$ for $m \geq 2$, use $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ from Example 3.1.6. A homomorphism $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Q}$ must send 1 to an element killed by m; the only such rational number is zero. A homomorphism to $\mathbb{Q}/\mathbb{Z}$ is instead determined by one of

$$0, \quad \tfrac{1}{m} + \mathbb{Z}, \quad \tfrac{2}{m} + \mathbb{Z}, \quad \dots, \quad \tfrac{m-1}{m} + \mathbb{Z}.$$

The resulting cochain complex is $0 \to 0 \to (\frac{1}{m}\mathbb{Z})/\mathbb{Z} \to 0$, with its two terms in degrees zero and one. Hence

$$\operatorname{Mor}_{\mathrm{Ab}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = 0, \qquad \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z},$$

and higher Ext vanishes. The projective-resolution calculation gives $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$, with the same kernel zero and cokernel $\mathbb{Z}/m\mathbb{Z}$. This checks the comparison by two distinct resolutions.

3.4.4 Comparing Tor and Ext Computations

For comparison with Section 3.2, suppose R is commutative and a is a non-zero-divisor. The same resolution used there now gives

$$0 \longrightarrow N \xrightarrow{a} N \longrightarrow 0 \qquad \text{in cochain degrees } 0, 1.$$

Hence $\operatorname{Mor}_{\mathrm{Mod}_R}(R/(a), N) = N[a]$, $\operatorname{Ext}^1_R(R/(a), N) = N/Na$, and higher Ext vanishes. The kernel and cokernel appear in the opposite degrees from the tensor computation. For the dual-number example, applying $\operatorname{Mor}_{\mathrm{Mod}_R}(-, K)$ makes every differential zero, so $\operatorname{Ext}^n_R(K, K) \cong k$ for every $n \geq 0$.

The two procedures can be compared directly:

	Tensor and Tor, R commutative	Morphism Groups and Ext, R arbitrary
Resolution	$P_\bullet \twoheadrightarrow M$	$P_\bullet \twoheadrightarrow M$
Complex	$P_\bullet \otimes_R N$	$\operatorname{Mor}_{\mathrm{Mod}_R}(P_\bullet, N)$
Free term $\bigoplus_{\lambda\in\Lambda} R$	$\bigoplus_{\lambda\in\Lambda} N$	$\prod_\Lambda N$
Degree zero	$M \otimes_R N$	$\operatorname{Mor}_{\mathrm{Mod}_R}(M, N)$
Higher degrees	$\mathrm{H}_i = \operatorname{Tor}^R_i(M, N)$	$\mathrm{H}^i = \operatorname{Ext}^i_R(M, N)$
Variance	covariant in both	contravariant, then covariant

In either computation, the central work is to construct enough of a resolution and then compute a kernel modulo an image. A single presentation suffices for a tensor product; its higher relations control Tor and Ext.

For further reading on tensor products and Tor, see [Stacks, Tensor Products, Tag 00CV; Tor Groups and Flatness, Tag 00LY].

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