Simplicial Sets, Spaces, and Singular Chains

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4.1 Simplicial Sets and Their Chains

In this section, R is commutative. Write Mod_R for the category of right R-modules; the usual identification $mr = rm$ identifies left and right modules over a commutative ring. A simplicial set records simplices and the ways they meet. Its simplices are initially elements of sets, so there is no addition and no alternating sum of faces. Passing to free modules introduces the algebra needed for a chain complex.

4.1.1 Simplicial Sets and Geometric Constructions

Definition 4.1.1. The simplex category Δ has objects $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$, and order-preserving maps as morphisms. A simplicial set is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$. Write $X_n = X([n])$. A map of simplicial sets is a natural transformation, and their category is $\mathcal{P}(\Delta)$.

Definition 4.1.2. The injection $[n-1] \to [n]$ omitting i induces a face map $d_i : X_n \to X_{n-1}$. The surjection $[n+1] \to [n]$ identifying i and $i+1$ induces a degeneracy map $s_i : X_n \to X_{n+1}$. These satisfy the simplicial identities

$$
\begin{aligned}
d_i d_j &= d_{j-1} d_i && (i < j),\\
s_i s_j &= s_{j+1} s_i && (i \leq j),\\
d_i s_j &= \begin{cases} s_{j-1} d_i, & i < j,\\ \mathrm{id}, & i = j \text{ or } i = j+1,\\ s_j d_{i-1}, & i > j+1. \end{cases}
\end{aligned}
$$

They express the equalities among order-preserving maps in Δ. A simplex in the image of a degeneracy is degenerate; otherwise it is nondegenerate.

Example 4.1.3. The standard simplicial n-simplex is the representable functor

$$\Delta^n_k = \mathrm{Mor}_{\Delta}([k], [n]).$$

Its boundary $\partial\Delta^n$ consists of the maps whose images miss at least one vertex. The quotient $\Delta^1/\partial\Delta^1$ is a simplicial circle: it has one nondegenerate vertex v and one nondegenerate edge e, with $d_0 e = d_1 e = v$, and no nondegenerate simplices in higher degrees. It still has degenerate simplices in every degree.

Definition 4.1.4. For a topological space Y, let

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0,\ \textstyle\sum_i t_i = 1\}.$$

The singular simplicial set $\mathrm{Sing}(Y)$ has continuous maps $|\Delta^n| \to Y$ as its n-simplices. Its structure maps are precomposition with the affine maps between topological simplices induced by maps in Δ.

For a simplicial set X, its geometric realization is

$$|X| = \left(\coprod_{n \geq 0} X_n \times |\Delta^n| \right) / \sim,$$

where $(X(\alpha)(x), t) \sim (x, |\alpha|(t))$ for $\alpha : [m] \to [n]$, $x \in X_n$, and $t \in |\Delta^m|$. Each X_n is given the discrete topology, and the quotient has the quotient topology.

The equivalence relation glues faces and collapses degenerate simplices. The definitions give the natural adjunction

$$\mathrm{Mor}_{\mathrm{Top}}(|X|, Y) \cong \mathrm{Mor}_{\mathcal{P}(\Delta)}(X, \mathrm{Sing}(Y)) :$$

a continuous map out of $|X|$ is exactly a compatible family of maps from its topological simplices. In Example 4.1.3, realization identifies the two endpoints of an interval, giving the usual circle.

4.1.2 Free Simplicial Modules and the Boundary

Definition 4.1.5. A simplicial R-module is a functor $A_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Mod}_R$. Its associated chain complex has $\mathrm{C}_n(A_\bullet) = A_n$ and differential

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i \quad (n > 0), \qquad \partial_0 = 0.$$

For a simplicial set X, let $R[X_n]$ be the free R-module on X_n. Extending the structure maps linearly gives a simplicial module $R[X]$, and hence a chain complex $\mathrm{C}_\bullet(X; R) = \mathrm{C}_\bullet(R[X])$.

Proposition 4.1.6. For a simplicial module, the alternating face differential satisfies $\partial_{n-1}\partial_n = 0$.

Proof. In the expanded double sum, pair the term $(-1)^{i+j} d_i d_j$ with $i < j$ with the term $(-1)^{i+j-1} d_{j-1} d_i$. The two composites agree by the face identity and have opposite signs. Every term belongs to one such pair. □

In dimension two this is the familiar calculation

$$\partial[0,1,2] = [1,2] - [0,2] + [0,1], \qquad \partial^2[0,1,2] = 0.$$

Taking $X = \mathrm{Sing}(Y)$ gives the usual singular chain complex, since its basis in degree n is precisely the set of singular n-simplices of Y.

4.1.3 Simplicial Chain Complexes in Low Degrees

For a simplicial abelian group $A_\bullet$, the construction is simply

$$\cdots \longrightarrow A_3 \xrightarrow{d_0 - d_1 + d_2 - d_3} A_2 \xrightarrow{d_0 - d_1 + d_2} A_1 \xrightarrow{d_0 - d_1} A_0 \longrightarrow 0.$$

All simplices are retained. The degeneracy maps do not occur in this formula for the differential, though they remain part of the simplicial structure.

It is useful to check the first nontrivial composition in full:

$$\begin{aligned}(d_0 - d_1)(d_0 - d_1 + d_2) &= d_0 d_0 - d_0 d_1 + d_0 d_2 \\ &\quad - d_1 d_0 + d_1 d_1 - d_1 d_2 = 0.\end{aligned}$$

The cancellations use $d_0 d_0 = d_0 d_1$, $d_0 d_2 = d_1 d_0$, and $d_1 d_1 = d_1 d_2$. Proposition 4.1.6 is the same cancellation in every degree.

A map of simplicial abelian groups $f : A_\bullet \to B_\bullet$ is a family of homomorphisms commuting with all face and degeneracy maps. In particular,

$$f_{n-1}\partial_n = \sum_i (-1)^i f_{n-1} d_i = \sum_i (-1)^i d_i f_n = \partial_n f_n.$$

Thus the associated chain complex construction is a functor: simplicial maps become chain maps, and hence induce maps on homology.

Example 4.1.7. Let G be an abelian group. Set $A_n = G$ in every degree and take every face and degeneracy map to be the identity. For $n > 0$ the differential is

$$\partial_n = \left(\sum_{i=0}^{n} (-1)^i\right) \mathrm{id}_G = \begin{cases} \mathrm{id}_G, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

The associated chain complex is therefore

$$\cdots \xrightarrow{0} G \xrightarrow{\mathrm{id}} G \xrightarrow{0} G \xrightarrow{\mathrm{id}} G \xrightarrow{0} G \longrightarrow 0,$$

where the rightmost G is in degree zero. It has $\mathrm{H}_0 = G$ and $\mathrm{H}_n = 0$ for every $n > 0$. A constant simplicial group is not sent to a chain complex with G only in degree zero; it is sent to a larger complex with exactly that homology.

Example 4.1.8. For a simplicial set X, take $A_n = \mathbb{Z}[X_n]$. An edge e has boundary $d_0 e - d_1 e$, its terminal vertex minus its initial vertex. Thus

$$\mathrm{H}_0(\mathrm{C}_\bullet(X;\mathbb{Z})) = \mathbb{Z}[X_0]/\langle d_0 e - d_1 e : e \in X_1 \rangle.$$

This is the free abelian group on the equivalence classes of vertices under the equivalence relation generated by the edges. These are the components of the simplicial set.

A triangle t has boundary $d_0 t - d_1 t + d_2 t$. Hence $\mathrm{H}_1(X;\mathbb{Z})$ is the group of integral sums of edges with zero total boundary, modulo boundaries of sums of triangles. In the simplicial circle of Example 4.1.3, the edge e is a cycle because both of its endpoints are v. Degenerate simplices are retained in all of these chain groups as well.

4.2 Presheaves, Simplices, and Simplicial Sets

The simplicial constructions of Section 4.1 are instances of functor categories. We now describe their natural transformations and universal properties.

4.2.1 Presheaves and the Yoneda Lemma

Definition 4.2.1. Let $\mathcal{D}$ be a small category, regarded as a diagram shape, and let $\mathcal{C}$ be a locally small category. The category of $\mathcal{C}$-valued presheaves on $\mathcal{D}$ is

$$\mathcal{P}(\mathcal{D};\mathcal{C}) = \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{C}).$$

Its objects are functors $F : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$, equivalently contravariant diagrams on $\mathcal{D}$. Thus F assigns an object $F(d)$ to each $d \in \mathcal{D}$ and a restriction map $f^* = F(f) : F(d') \to F(d)$ to each $f : d \to d'$, with

$$\mathrm{id}_d^* = \mathrm{id}_{F(d)}, \qquad (g \circ f)^* = f^* \circ g^*.$$

Its morphisms are natural transformations: a morphism $\eta : F \to G$ is a family $\eta_d : F(d) \to G(d)$ satisfying

$$G(f) \circ \eta_{d'} = \eta_d \circ F(f) \qquad \text{for every } f : d \to d'.$$

Identities and composition are componentwise: $(\mathrm{id}_F)_d = \mathrm{id}_{F(d)}$ and $(\theta \circ \eta)_d = \theta_d \circ \eta_d$.

The first argument of $\mathcal{P}(\mathcal{D};\mathcal{C})$ specifies the diagram shape; the second specifies the category of values. The opposite category $\mathcal{D}^{\mathrm{op}}$ reverses arrows. Here small means that objects and morphisms form sets; locally small means that morphisms between any two fixed objects form a set.

More generally, the functor category $\mathrm{Fun}(\mathcal{I},\mathcal{C})$ has covariant functors $\mathcal{I} \to \mathcal{C}$ as objects and natural transformations as morphisms. For $f : i \to j$, naturality reads $G(f)\eta_i = \eta_j F(f)$; composition is again componentwise. The definition above is the case $\mathcal{I} = \mathcal{D}^{\mathrm{op}}$.

Definition 4.2.2. Specializing to $\mathcal{C} = \mathrm{Set}$, write

$$\mathcal{P}(\mathcal{D}) = \mathcal{P}(\mathcal{D}; \mathrm{Set}) = \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Set}).$$

Thus an omitted second argument means sets. By contrast, $\mathcal{P}(\mathcal{D}; \mathrm{Ab})$ consists of contravariant diagrams of abelian groups and homomorphisms.

Each object $d \in \mathcal{D}$ defines the representable presheaf $\mathrm{Mor}_{\mathcal{D}}(-, d)$, whose value at e is $\mathrm{Mor}_{\mathcal{D}}(e, d)$ and whose restriction maps are precomposition. Simplicial sets are the case $\mathcal{D} = \Delta$.

Proposition 4.2.3. For $F \in \mathcal{P}(\mathcal{D})$ and $d \in \mathcal{D}$, evaluation at id_d gives a natural bijection

$$\mathrm{Mor}_{\mathcal{P}(\mathcal{D})}(\mathrm{Mor}_{\mathcal{D}}(-, d), F) \cong F(d).$$

Proof. An element $x \in F(d)$ defines $\eta_e(f) = f^* x$ for $f : e \to d$. Functoriality makes η natural, and naturality forces this formula from $x = \eta_d(\mathrm{id}_d)$. These constructions are inverse and natural. □

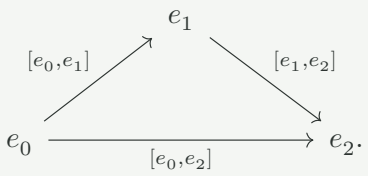
4.2.2 Topological Simplices and Affine Operators

Definition 4.2.4. Use the standard topological simplex of Definition 4.1.4. Its vertices are the coordinate vectors $e_0, \dots, e_n$, and its coordinates t_i are called barycentric coordinates. The realization notation will be justified below.

Thus $|\Delta^0|$ is a point, $|\Delta^1|$ is an interval, and $|\Delta^2|$ is a filled triangle. The ordering of the vertices determines the signs in its oriented boundary:

$$\partial[e_0, e_1, e_2] = [e_1, e_2] - [e_0, e_2] + [e_0, e_1].$$

The triangle can be displayed as



This is an oriented geometric triangle. In a general simplicial set, its edges are not assumed to be morphisms in a category, and the triangle is not an assertion of equality between composites.

Definition 4.2.5. For the simplex category Δ of Definition 4.1.1, a map $\alpha : [m] \to [n]$ determines the affine map

$$\alpha_* : |\Delta^m| \longrightarrow |\Delta^n|, \qquad (\alpha_*(t))_j = \sum_{\alpha(i)=j} t_i.$$

An empty sum is zero. In particular, $\alpha_*(e_i) = e_{\alpha(i)}$.

For $\beta : [\ell] \to [m]$, grouping the coordinates gives $(\alpha\beta)_* = \alpha_* \beta_*$. Thus we obtain a covariant functor $\Delta \to \mathrm{Top}$ that sends $[n]$ to $|\Delta^n|$.

Two kinds of maps generate Δ:

1. The coface $d^i : [n-1] \hookrightarrow [n]$ omits i. Its affine map inserts a zero into barycentric coordinates at position i.
2. The codegeneracy $\sigma^i : [n+1] \twoheadrightarrow [n]$ identifies i and $i+1$. Its affine map replaces the adjacent coordinates t_i, t_{i+1} by their sum.

Every order-preserving map factors uniquely as a surjection followed by an injection. Geometrically, it first collapses repeated vertices and then includes a face.

4.2.3 Simplicial Sets

Definition 4.2.6. The simplicial sets of Definition 4.1.1, form the presheaf category $\mathcal{P}(\Delta) = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$. For a simplicial set K, write $K_n = K([n])$ and $\alpha^* = K(\alpha)$ for its restriction maps.

The constructions fit into the following diagram of functors:

$$\mathrm{Top} \underset{|-|}{\overset{\mathrm{Sing}}{\rightleftarrows}} \mathcal{P}(\Delta) \underset{U}{\overset{\mathbb{Z}[-]}{\rightleftarrows}} \mathcal{P}(\Delta;\mathrm{Ab}) \xrightarrow{\mathrm{C}^{\mathrm{alt}}} \mathrm{Ch}_{\geq 0}(\mathrm{Ab}).$$

Here $|-| \dashv \mathrm{Sing}$ and $\mathbb{Z}[-] \dashv U$: realization is left adjoint to the singular functor, and the free simplicial abelian group functor is left adjoint to forgetting addition. The composite from left to right is the singular chain functor.

Elements of K_n are called n-simplices. The following theorem gives an equivalent description using only faces and degeneracies.

Theorem 4.2.7. The data of a simplicial set is equivalently a sequence of sets $K_0, K_1, K_2, \ldots$ equipped with face maps and degeneracy maps

$$\begin{aligned} d_i &: K_n \longrightarrow K_{n-1} \quad (n \geq 1,\ 0 \leq i \leq n), \\ s_i &: K_n \longrightarrow K_{n+1} \quad (n \geq 0,\ 0 \leq i \leq n), \end{aligned}$$

subject to the simplicial identities

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && (i < j), \\ s_i s_j &= s_{j+1} s_i && (i \leq j), \\ d_i s_j &= \begin{cases} s_{j-1} d_i, & i < j, \\ \mathrm{id}, & i = j \text{ or } i = j+1, \\ s_j d_{i-1}, & i > j+1. \end{cases} \end{aligned}$$

The identities are required whenever the indicated composites are defined; composition is read from right to left.

More precisely, every such system extends uniquely to a functor $K : \Delta^{\mathrm{op}} \to \mathrm{Set}$ with $K([n]) = K_n$, $K(d^i) = d_i$, and $K(\sigma^i) = s_i$. Under this correspondence, simplicial maps are exactly families of functions $f_n : K_n \to L_n$ commuting with every face and degeneracy map.

Proof. A functor gives the stated maps by applying it to cofaces and codegeneracies. Their relations in Δ, with composition reversed, give the displayed identities.

Conversely, let $\alpha : [m] \to [n]$ have image of size $r+1$. Its unique factorization is $\alpha = \iota\pi$, where $\pi : [m] \twoheadrightarrow [r]$ records the position of each value in the ordered image and $\iota : [r] \hookrightarrow [n]$ includes that image. If $a_1 < \cdots < a_k$ are the omitted target values and $b_1 < \cdots < b_\ell$ are the indices with $\alpha(b_j) = \alpha(b_j + 1)$, its canonical expression is

$$\alpha = d^{a_k} \cdots d^{a_1} \sigma^{b_1} \cdots \sigma^{b_\ell}, \qquad k = n - r, \quad \ell = m - r.$$

An empty string denotes an identity. Each factor is given the domain required by the next factor to its right. Deleting repeated positions from largest to smallest proves the formula for π; inserting omitted positions from smallest to largest proves the formula for ι.

For completeness, the relations give a terminating reduction to this expression. A mixed adjacent pair consisting of a codegeneracy to the left of a coface is rewritten as a coface followed by a codegeneracy, or cancelled. At fixed word length each such interchange reduces the number of pairs in the wrong order; cancellation reduces length. Thus all cofaces can

be placed to the left. Within the two blocks, their elementary relations order the omitted and repeated indices as displayed. This last ordering can be performed by insertion: insert each new generator into an already ordered block, moving it across successive neighbors and adjusting its index by the relation. The block is finite, so each insertion terminates. An ordered block is uniquely determined by its omitted values or repeated positions. Hence two words for the same map reduce to the same expression. These relations therefore present Δ by generators and relations.

Assigning d_i and s_i to these generators therefore defines the pullback along every map of Δ, independently of its expression. The same relations ensure functoriality. Generation gives uniqueness and also shows that commuting with faces and degeneracies is equivalent to naturality. For the generators-and-relations formulation, see [Stacks, Lemmas 14.2.4 and 14.3.2], in the sections *The Category of Finite Ordered Sets* and *Simplicial Objects*. □

The faces of an edge e are its target $d_0 e$ and source $d_1 e$. The simplex $s_0 v$ is the degenerate edge at v; both of its endpoints are v. Geometrically, the first identity says that deleting vertex j and then vertex $i < j$ has the same effect as deleting vertex i and then the renumbered vertex $j-1$.

A simplex is *degenerate* if it is in the image of some s_i; otherwise it is *nondegenerate*. A degenerate simplex is still an element of K_n. It is not removed from the simplicial set.

Example 4.2.8. Use the representable standard simplex Δ^n of Example 4.1.3. Precomposition gives $\beta^*(\alpha) = \alpha\beta$. Its nondegenerate m-simplices are precisely the injective order-preserving maps $[m] \hookrightarrow [n]$. Thus Δ^1 has two nondegenerate vertices and one nondegenerate edge, while retaining simplices in every degree.

The notations $[n]$, Δ^n, and $|\Delta^n|$ refer respectively to an ordered set, a simplicial set, and a topological space.

Proposition 4.2.9. There is a natural bijection

$$\mathrm{Mor}_{\mathcal{P}(\Delta)}(\Delta^n, K) \cong K_n.$$

Proof. The standard simplex Δ^n is the representable presheaf $\mathrm{Mor}_\Delta(-,[n])$. Apply Yoneda: $x \in K_n$ corresponds to the simplicial map $\Delta^n \to K$ whose degree-m component is $\mathrm{Mor}_\Delta([m],[n]) \to K_m$, $\alpha \mapsto \alpha^* x$. □

4.3 Geometric Realization and Singular Simplices

4.3.1 Geometric Realization

The construction is discussed in [Kerodon, Section 1.2.3], *The Geometric Realization of a Simplicial Set*, especially Proposition 1.2.3.4 and Corollary 1.2.3.5. The realization–singular bijection is proved explicitly in Subsection 4.3.3.

Definition 4.3.1. Use the quotient realization $|K|$ of Definition 4.1.4, with its quotient topology. In the notation of Definition 4.2.5, its relations are $(\alpha^* x, t) \sim (x, \alpha_*(t))$. Write $[x,t]$ for the equivalence class of (x,t).

The face relations attach the boundary of one simplex to the specified lower-dimensional simplices. The degeneracy relations collapse a redundant simplex onto its lower-dimensional source. For example,

$$[s_0 v, (t_0, t_1)] = [v, (1)],$$

so a degenerate edge becomes a point, not an extra loop.

Every $x \in K_n$ has a characteristic map

$$\chi_x : |\Delta^n| \longrightarrow |K|, \qquad t \longmapsto [x, t].$$

These maps obey the commutative diagrams

$$\begin{array}{ccc} |\Delta^m| & \xrightarrow{\alpha_*} & |\Delta^n| \\ {\scriptstyle \chi_{\alpha^* x}}\downarrow & & \downarrow{\scriptstyle \chi_x} \\ |K| & = & |K|. \end{array}$$

Define the category of simplices Δ/K to have objects $([n], x)$ with $x \in K_n$; a morphism $([m], y) \to ([n], x)$ is a map $\alpha : [m] \to [n]$ with $y = \alpha^* x$. Then

$$|K| \cong \operatorname*{colim}_{([n],x) \in \Delta/K} |\Delta^n|.$$

This is a colimit in Top: continuous maps $|K| \to X$ correspond exactly to families of continuous maps $f_x : |\Delta^n| \to X$ satisfying $f_{\alpha^* x} = f_x \alpha_*$. Indeed, such a family defines a continuous map from the disjoint union, and compatibility is precisely what makes it descend through the quotient.

A simplicial map $f : K \to L$ induces

$$|f| : |K| \to |L|, \qquad [x, t] \longmapsto [f_n(x), t].$$

Compatibility with the relation follows from $f_m(\alpha^* x) = \alpha^* f_n(x)$. This defines a functor $|-| : \mathcal{P}(\Delta) \to \mathrm{Top}$.

Example 4.3.2.

1. The realization of the standard simplicial simplex is canonically the topological simplex introduced in Subsection 4.2.2. The map $|\Delta^n| \to \{t \in \mathbb{R}^{n+1} : t_i \geq 0,\ \sum_i t_i = 1\}$ is given by $[\alpha, t] \mapsto \alpha_*(t)$. Its inverse sends u to $[\mathrm{id}_{[n]}, u]$; the defining relation proves that the two composites are identities. We use $|\Delta^n|$ for this identified space.
2. A constant simplicial set on a set S has value S and every structure map is the identity. Its realization is the discrete space S.
3. Write S^1_Δ for the simplicial circle $\Delta^1/\partial\Delta^1$ of Example 4.1.3. Its realization was identified there with the topological circle.

Repeated vertices, loops, and distinct simplices with the same boundary are all allowed in a simplicial set. This flexibility goes beyond an abstract simplicial complex.

4.3.2 Singular Simplices

Definition 4.3.3. Use the singular simplicial set of Definition 4.1.4. We work in Top, without separation assumptions, and write $(\mathrm{Sing}\, X)_n = \mathrm{Mor}_{\mathrm{Top}}(|\Delta^n|, X)$. In the affine notation of Definition 4.2.5, its restriction maps are $\alpha^*(\tau) = \tau\alpha_*$. These morphism sets carry no additional topology here.

Precomposition and $(\alpha\beta)_* = \alpha_*\beta_*$ give the required contravariance. Thus $\mathrm{Sing}\, X$ is a simplicial set, called the *singular simplicial set* of X.

The first few degrees have direct geometric meanings:

1. $(\mathrm{Sing}\, X)_0$ is the underlying set of points of X.

2. $(\operatorname{Sing} X)_1$ is the set of continuous paths, with the interval parametrized by $(1-u, u)$ for $0 \le u \le 1$. Thus $d_1\gamma = \gamma(0)$ and $d_0\gamma = \gamma(1)$.
3. $(\operatorname{Sing} X)_2$ consists of continuous maps of a filled triangle into X. Its three faces are the restrictions to its three ordered edges.

A singular simplex need not be injective or linear, and its image need not look like a geometric simplex. In particular, an arbitrary space need not come with a chosen triangulation for its singular simplicial set to be defined.

For $\tau : |\Delta^n| \to X$, the degeneracy $s_i\tau : |\Delta^{n+1}| \to X$ is

$$(t_0, \dots, t_{n+1}) \longmapsto \tau(t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}).$$

For $n = 0$, this is a constant path. A path that goes out and comes back is generally *not* a degenerate edge: degeneracy means the specified factorization through a lower-dimensional standard simplex, not merely that endpoints coincide.

If $f : X \to Y$ is continuous, postcomposition defines

$$(\operatorname{Sing} f)_n : (\operatorname{Sing} X)_n \to (\operatorname{Sing} Y)_n, \qquad \tau \longmapsto f\tau.$$

Since postcomposition commutes with precomposition, this is simplicial and defines the functor $\operatorname{Sing} : \mathrm{Top} \to \mathcal{P}(\Delta)$.

Example 4.3.4. The singular simplicial set of a point is Δ^0: it has one simplex in every degree. If X is discrete, every map $|\Delta^n| \to X$ is constant, so $\operatorname{Sing} X$ is the constant simplicial set on X. For $X = [0, 1]$, the simplicial set $\operatorname{Sing} X$ is much larger than Δ^1: it includes every continuous path and every continuous singular simplex in the interval.

4.3.3 The Realization–Singular Adjunction

An adjunction here is a natural bijection between two sets of maps. The relevant bijection says that mapping a glued space into X is the same as assigning compatible singular simplices in X to its pieces.

Theorem 4.3.5. Use the realization–singular adjunction established after Definition 4.1.4. The proof below expresses its bijection through the colimit universal property, preparing the unit and counit construction.

Proof. By the colimit universal property, a continuous map $|K| \to X$ is a compatible family $f_x : |\Delta^n| \to X$. Since $(\operatorname{Sing} X)_n = \mathrm{Mor}_{\mathrm{Top}}(|\Delta^n|, X)$, this is exactly a family of functions

$$K_n \longrightarrow (\operatorname{Sing} X)_n, \qquad x \longmapsto f_x,$$

commuting with restriction maps: a morphism of presheaves $K \to \operatorname{Sing} X$. Explicitly, $f : |K| \to X$ determines $g : K \to \operatorname{Sing} X$ with $g_n(x) = f\chi_x$. Conversely, $g : K \to \operatorname{Sing} X$ determines $f : |K| \to X$ by $f([x, t]) = g_n(x)(t)$. Both commute with precomposition in K and postcomposition in X, so the bijection is natural. □

The identity of $|K|$ corresponds to the *unit*

$$(\eta_K)_n : K_n \longrightarrow (\operatorname{Sing} |K|)_n, \qquad x \longmapsto \chi_x.$$

Thus each abstract simplex becomes the singular simplex that parametrizes its own image in the realization.

The identity of $\operatorname{Sing} X$ corresponds to the *counit*, or evaluation map,

$$\epsilon_X : |\operatorname{Sing} X| \longrightarrow X, \qquad [\tau, t] \longmapsto \tau(t).$$

The triangle identities are

$$\epsilon_{|K|} \circ |\eta_K| = \mathrm{id}_{|K|}, \qquad \mathrm{Sing}(\epsilon_X) \circ \eta_{\mathrm{Sing}\, X} = \mathrm{id}_{\mathrm{Sing}\, X}.$$

For example, the first composite sends $[x,t]$ to $[\chi_x, t]$ and then to $\chi_x(t) = [x,t]$. The second says that evaluating the characteristic simplex associated to τ recovers τ.

Remark 4.3.6. The map η_{Δ^1} includes the simplices determined by the ordered vertices into all singular simplices of the interval; most continuous paths are not in its image. Likewise, evaluation need not be a homeomorphism. The deeper comparison with homotopy types is a theorem beyond the adjunction itself, and is not needed to construct singular chains. For further context on this adjunction and its relation to cellular homology, see [May99, Chapter 16, §4, pp. 126–127], where the chain complexes are normalized.

4.4 Singular Chains and Coefficients

4.4.1 The Free Simplicial Abelian Group Functor

Definition 4.4.1. For a set S, the free abelian group on S is

$$\mathbb{Z}[S] = \bigoplus_{s \in S} \mathbb{Z} s.$$

Its elements are finite formal sums $\sum_s a_s s$ with $a_s \in \mathbb{Z}$. For a function $f : S \to T$, set

$$\mathbb{Z}[f]\left(\sum_s a_s s\right) = \sum_s a_s f(s).$$

We write s also for the formal generator labelled by $s \in S$; addition in $\mathbb{Z}[S]$ is addition of formal sums. If f identifies two elements, their coefficients add. For example, if $f(s) = f(s') = t$, then $2s - s'$ maps to t.

For every abelian group B, a function $S \to U(B)$ extends uniquely to a homomorphism $\mathbb{Z}[S] \to B$ by linearity. Here $U(B)$ denotes its underlying set. Thus

$$\mathrm{Mor}_{\mathsf{Ab}}(\mathbb{Z}[S], B) \cong \mathrm{Mor}_{\mathsf{Set}}(S, U(B)).$$

Definition 4.4.2. The simplicial modules of Definition 4.1.5, specialize over $\mathbb{Z}$ to simplicial abelian groups. They form $\mathcal{P}(\Delta; \mathsf{Ab})$. Forgetting their group structures degreewise defines $U : \mathcal{P}(\Delta; \mathsf{Ab}) \to \mathcal{P}(\Delta)$.

For a simplicial set K, define

$$\mathbb{Z}[K]_n = \mathbb{Z}[K_n], \qquad \alpha^*\left(\sum_x a_x x\right) = \sum_x a_x \alpha^* x.$$

The simplicial identities hold on generators because they hold in K, and therefore hold on all finite sums. In particular,

$$d_i x = d_i x, \qquad s_i x = s_i x.$$

A simplicial map $f : K \to L$ induces the degreewise homomorphisms $\mathbb{Z}[f_n] : \mathbb{Z}[K_n] \to \mathbb{Z}[L_n]$ given by $x \mapsto f_n(x)$. We obtain the functor

$$\mathbb{Z}[-] : \mathcal{P}(\Delta) \longrightarrow \mathcal{P}(\Delta; \mathsf{Ab}).$$

The group in degree n is free on *all* n-simplices, including degenerate simplices. It is not the free group on the vertices alone.

Proposition 4.4.3. Let $L : \mathcal{D} \rightleftarrows \mathcal{E} : R$ be an adjunction and $\mathcal{I}$ a small category. Postcomposition induces an adjunction

$$L_* : \mathrm{Fun}(\mathcal{I}, \mathcal{D}) \rightleftarrows \mathrm{Fun}(\mathcal{I}, \mathcal{E}) : R_*.$$

Proof. For $F : \mathcal{I} \to \mathcal{D}$ and $G : \mathcal{I} \to \mathcal{E}$, transpose each component $LF(i) \to G(i)$ under $L \dashv R$. Naturality of the adjunction bijection identifies the naturality equations for these components with those for $F(i) \to RG(i)$. Thus componentwise transposition gives a natural bijection between the two sets of natural transformations. □

Proposition 4.4.4. For $K \in \mathcal{P}(\Delta)$ and $A \in \mathcal{P}(\Delta; \mathrm{Ab})$, there is a natural bijection

$$\mathrm{Mor}_{\mathcal{P}(\Delta;\mathrm{Ab})}(\mathbb{Z}[K], A) \cong \mathrm{Mor}_{\mathcal{P}(\Delta)}(K, U(A)).$$

Proof. Apply the pointwise adjunction proposition to $\mathbb{Z}[-] : \mathrm{Set} \rightleftarrows \mathrm{Ab} : U$ with $\mathcal{I} = \Delta^{\mathrm{op}}$. Concretely, $g : K \to U(A)$ extends uniquely by

$$\overline{g}_n \left(\sum_x a_x x \right) = \sum_x a_x g_n(x).$$

Naturality of g makes these extensions commute with every simplicial operator. Restriction to generators is the inverse. □

The unit in degree n is the function $K_n \to \mathbb{Z}[K_n]$ sending a simplex to its formal generator. The counit is the simplicial homomorphism $\mathbb{Z}[U(A)] \to A$ that evaluates each finite formal sum using the addition in A. A generator labelled by a sum in A is distinct from the formal sum of the corresponding generators; the counit sends them to the same element.

Remark 4.4.5. The functor $\mathbb{Z}[-]$ freely introduces sums and additive inverses while retaining all faces and degeneracies. A simplicial abelian group still has $n+1$ face maps in degree n. The next construction combines them into one differential.

4.4.2 Alternating Faces and Degenerate Simplices

Definition 4.4.6. Use the alternating-face complex of Definition 4.1.5, retaining degenerate simplices. Here $\mathrm{C}^{\mathrm{alt}}(A)$ denotes that construction for simplicial abelian groups, with homological grading and zero groups in negative degrees.

The first two differentials are $\partial_1 = d_0 - d_1$ and $\partial_2 = d_0 - d_1 + d_2$. Thus the boundary of an edge is target minus source, and the boundary of a triangle is its three edges with alternating signs.

Proposition 4.4.7. Proposition 4.1.6 and Section 4.1.3 prove functoriality of the alternating-face construction:

$$\mathrm{C}^{\mathrm{alt}} : \mathcal{P}(\Delta; \mathrm{Ab}) \longrightarrow \mathrm{Ch}_{\geq 0}(\mathrm{Ab}).$$

Proof. The face-pairing proof and the compatibility with simplicial maps are given in Proposition 4.1.6 and Section 4.1.3. □

Each codimension-two face occurs twice, with opposite signs. See also [Stacks, Section 14.23, Simplicial Objects and Chain Complexes]; our notation $\mathrm{C}^{\mathrm{alt}}(A)$ denotes the alternating-face complex called $s(A)$ there.

For a simplicial set K, write

$$\mathrm{C}_\bullet(K; \mathbb{Z}) = \mathrm{C}^{\mathrm{alt}}(\mathbb{Z}[K]), \qquad \mathrm{H}_n(K; \mathbb{Z}) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1}).$$

These are the unnormalized simplicial chains and their homology. In positive degrees,

$$\partial x = \sum_{i=0}^{n} (-1)^i d_i x.$$

A degenerate simplex is not the zero element of a free chain group. For instance, the constant edge $\alpha = s_0 v$ has $\partial\alpha = 0$, but α is a nonzero generator of C_1. It becomes a boundary: if $\beta = s_0 s_0 v$, then all three faces of β equal α, so $\partial\beta = \alpha - \alpha + \alpha = \alpha$.

Example 4.4.8. Apply the constant simplicial abelian-group computation of Example 4.1.7, to $G = \mathbb{Z}$. Since $\mathbb{Z}[\Delta^0]$ is constant, this gives $\mathrm{H}_0(\Delta^0; \mathbb{Z}) = \mathbb{Z}$ and zero homology in positive degrees, while retaining degenerate simplices.

4.4.3 The Singular Chain Functor

Definition 4.4.9. The singular chain complex of a topological space X is

$$\mathrm{C}_\bullet^{\mathrm{sing}}(X; \mathbb{Z}) = \mathrm{C}^{\mathrm{alt}}(\mathbb{Z}[\mathrm{Sing}\, X]).$$

Its degree-n group is freely generated by continuous maps $\tau : |\Delta^n| \to X$. For $n \geq 1$, its boundary is

$$\partial\tau = \sum_{i=0}^{n} (-1)^i \tau d^i_*.$$

The singular homology of X is $\mathrm{H}_n^{\mathrm{sing}}(X; \mathbb{Z}) = \mathrm{H}_n(\mathrm{C}_\bullet^{\mathrm{sing}}(X; \mathbb{Z}))$.

The construction separates three steps:

1. Sing records all continuous simplices and their restrictions and repetitions.
2. $\mathbb{Z}[-]$ permits finite integral combinations of these simplices.
3. $\mathrm{C}^{\mathrm{alt}}$ adds the face maps with signs to define a boundary operator.

None of these steps requires choosing a triangulation of X.

For a continuous map $f : X \to Y$, the induced chain map sends τ to $f\tau$. For every $n \geq 1$, we have the commutative square

$$\begin{array}{ccc} \mathrm{C}_n^{\mathrm{sing}}(X; \mathbb{Z}) & \xrightarrow{f_*} & \mathrm{C}_n^{\mathrm{sing}}(Y; \mathbb{Z}) \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ \mathrm{C}_{n-1}^{\mathrm{sing}}(X; \mathbb{Z}) & \xrightarrow[f_*]{} & \mathrm{C}_{n-1}^{\mathrm{sing}}(Y; \mathbb{Z}). \end{array}$$

It follows that singular homology is functorial.

Example 4.4.10. For a path $\gamma : x \to y$, its boundary is $y - x$. Consequently,

$$\mathrm{H}_0^{\mathrm{sing}}(X; \mathbb{Z}) \cong \mathbb{Z}[\pi_0^{\mathrm{path}} X],$$

where $\pi_0^{\mathrm{path}} X$ is the set of path components, not merely connected components.

For a singular triangle τ with ordered edges $\gamma_{01}, \gamma_{02}, \gamma_{12}$,

$$\partial\tau = \gamma_{12} - \gamma_{02} + \gamma_{01}.$$

Thus the cycle given by the two successive edges minus the long edge is a boundary. This is a relation among chains modulo boundaries; the individual nonclosed edges need not represent classes in $\mathrm{H}_1^{\mathrm{sing}}(X; \mathbb{Z})$.

4.4.4 Coefficients and the Real Projective Plane

Definition 4.4.11. Let X be a simplicial set and let G be an abelian group. With $A_\bullet = \mathrm{C}_\bullet(X;\mathbb{Z})$, define

$$\mathrm{H}_n(X;G) = \mathrm{H}_n(A_\bullet \otimes_{\mathbb{Z}} G).$$

For a space Y, apply this construction to $X = \mathrm{Sing}(Y)$ to obtain singular homology with coefficients in G. The chain group is the direct sum of one copy of G for each singular simplex, and its differential is the alternating sum of restrictions to faces.

Example 4.4.12. This is a forward example: the topological input will be proved in Lecture 8, Theorems 8.2.2 and 8.3.4 (cellular comparison and the differential for real projective space). Appendix B, Proposition B.1.2 and the projective-plane calculation immediately following it, also explain the coefficient 2 from the attaching word aa. Using those results, the cellular chain complex of $\mathbb{RP}^2$ over $\mathbb{Z}$ is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

in degrees $2, 1, 0$. Thus $\mathrm{H}_1(\mathbb{RP}^2;\mathbb{Z}) = \mathbb{Z}/2$ and $\mathrm{H}_2(\mathbb{RP}^2;\mathbb{Z}) = 0$. The cellular comparison theorem identifies these groups with singular homology.

For coefficients $\mathbb{Z}/2$, the multiplication-by-two differential vanishes, giving

$$\mathrm{H}_2(\mathbb{RP}^2;\mathbb{Z}/2) \cong \mathbb{Z}/2.$$

This class is invisible in $\mathrm{H}_2(\mathbb{RP}^2;\mathbb{Z}) \otimes \mathbb{Z}/2$, which is zero. Its value agrees with

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}_1(\mathbb{RP}^2;\mathbb{Z}), \mathbb{Z}/2) = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

A simplicial chain complex is free in each degree, but it is usually not a resolution of its degree-zero homology: it may have higher homology. To compute Tor of a single module, use an actual resolution. To compute homology with coefficients of a space, retain the full chain complex of that space.

References

[Kerodon] *Kerodon*. https://kerodon.net.

[Stacks] The Stacks Project Authors, *The Stacks Project*, 2026. https://stacks.math.columbia.edu.

[May99] J. P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press, 1999. Further background for Lectures 5 and 7: Chapters 6, 10, and 13–15; Chapter 10, §7, p. 80 treats the double mapping cylinder.