Generalized Homology and the Eilenberg–Steenrod Axioms

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5.1 Homotopy Invariance and Homotopy Pushouts

The first part of a homology theory is functorial data. The gluing axiom needs a construction that remembers the homotopy along which two maps agree.

5.1.1 Graded Functors

For a space W, write $\mathrm{pr}_W : W \longrightarrow *$ for its unique map to a point.

Write Top for the category of topological spaces and continuous maps, and Ab for the category of abelian groups. Homotopy pushouts are understood up to homotopy equivalence.

Definition 5.1.1. A graded homology functor is a family of covariant functors

$$A_n : \mathrm{Top} \longrightarrow \mathrm{Ab} \qquad (n \in \mathbb{Z}).$$

For a map $f : X \longrightarrow Y$, write $f_* = A_n(f) : A_n(X) \longrightarrow A_n(Y)$. Functoriality means that identities induce identities and $(gf)_* = g_* f_*$. The degree is specified by the source and target when it is omitted from f_*.

5.1.2 Homotopies and Homotopy Invariance

Definition 5.1.2. Write $I = [0,1]$. Given maps

$$f_0, f_1 : X \longrightarrow Y,$$

a *homotopy* from f_0 to f_1 is a continuous map

$$\alpha : X \times I \longrightarrow Y, \qquad \alpha(x,0) = f_0(x), \quad \alpha(x,1) = f_1(x).$$

The endpoint inclusions are

$$\iota_0 : X \longrightarrow X \times I, \qquad x \longmapsto (x,0),$$
$$\iota_1 : X \longrightarrow X \times I, \qquad x \longmapsto (x,1).$$

Thus the endpoint conditions say that the following triangles commute:

$$\begin{array}{ccccc} X & \xrightarrow{\iota_0} & X \times I & \xleftarrow{\iota_1} & X \\ & \searrow^{f_0} & \downarrow{\scriptstyle \alpha} & \swarrow_{f_1} & \\ & & Y & & \end{array}$$

We write $f_0 \simeq f_1$. Constant homotopies, reversal of I, and concatenation show that this is an equivalence relation on the set of maps between two fixed spaces.

Definition 5.1.3. The category hTop has spaces as objects and morphism sets

$$\mathrm{Mor}_{\mathrm{hTop}}(X,Y) = \mathrm{Mor}_{\mathrm{Top}}(X,Y)/\simeq .$$

The class of a map f is written $[f]$. For composable maps

$$f : X \longrightarrow Y, \qquad g : Y \longrightarrow Z,$$

composition and identities are defined by

$$[g] \circ [f] = [g \circ f], \qquad \mathrm{id}_X = [\mathrm{id}_X].$$

To check composition, take homotopies

$$\alpha : X \times I \longrightarrow Y, \qquad \beta : Y \times I \longrightarrow Z.$$

The composite homotopy is the continuous map

$$\gamma : X \times I \longrightarrow Z, \qquad (x,t) \longmapsto \beta(\alpha(x,t),t).$$

Its endpoints are the composites of the endpoint maps of α and β. Hence composition of homotopy classes is well defined. Associativity and the identity laws follow from those in Top.

The quotient functor is

$$q : \mathrm{Top} \longrightarrow \mathrm{hTop}, \qquad q(X) = X, \quad q(f) = [f].$$

Definition 5.1.4. A *homotopy equivalence* is a map admitting a homotopy inverse:

$$f : X \longrightarrow Y, \qquad g : Y \longrightarrow X, \qquad gf \simeq \mathrm{id}_X, \quad fg \simeq \mathrm{id}_Y.$$

Thus f is a homotopy equivalence exactly when $[f]$ is an isomorphism in hTop, with inverse $[g]$.

Definition 5.1.5. Let $\mathcal{C}$ be a category. A functor

$$F : \mathrm{Top} \longrightarrow \mathcal{C}$$

is *homotopy invariant* if homotopic maps have equal images: for

$$f_0, f_1 : X \longrightarrow Y, \qquad f_0 \simeq f_1,$$

one has

$$F(f_0) = F(f_1) : F(X) \longrightarrow F(Y).$$

Equivalently, there is a unique functor

$$\overline{F} : \mathrm{hTop} \longrightarrow \mathcal{C}, \qquad \overline{F}(X) = F(X), \quad \overline{F}([f]) = F(f),$$

making the triangle commute:

$$\begin{array}{ccc} \mathrm{Top} & \xrightarrow{\ q\ } & \mathrm{hTop} \\ & \searrow^{F} & \downarrow{\scriptstyle \overline{F}} \\ & & \mathcal{C}. \end{array}$$

Homotopy invariance is exactly what makes the displayed formula on morphisms well defined. The identity and composition laws descend from F, and the formulas force uniqueness.

Proposition 5.1.6. Every homotopy invariant functor sends homotopy equivalences to isomorphisms.

Proof. For a homotopy equivalence and its homotopy inverse, write

$$f : X \longrightarrow Y, \qquad g : Y \longrightarrow X.$$

Their images have types

$$F(f) : F(X) \longrightarrow F(Y), \qquad F(g) : F(Y) \longrightarrow F(X).$$

Functoriality and homotopy invariance give

$$F(g)F(f) = F(gf) = \mathrm{id}_{F(X)}, \qquad F(f)F(g) = F(fg) = \mathrm{id}_{F(Y)}.$$

Thus $F(g)$ is the inverse of $F(f)$. □

5.1.3 Homotopy Pushouts

The gluing axiom uses the double mapping cylinder to retain the homotopy between the two composites.

Definition 5.1.7. For a span $X \xleftarrow{f} W \xrightarrow{g} Y$, its double mapping cylinder is

$$M(f,g) = (X + (W \times I) + Y)/\sim,$$

where $(w,0) \sim f(w)$ and $(w,1) \sim g(w)$. Here $+$ denotes a finite disjoint union. Let $\iota_X : X \longrightarrow M(f,g)$ and $\iota_Y : Y \longrightarrow M(f,g)$ be the canonical maps. The cylinder supplies a specified homotopy from $\iota_X f$ to $\iota_Y g$.

A map from $M(f,g)$ to a space Z is equivalent to maps $h : X \longrightarrow Z$, $k : Y \longrightarrow Z$, and a homotopy from hf to kg. This follows from the quotient universal property: the homotopy defines the map on $W \times I$, and its endpoint conditions are precisely the identifications in the quotient.

Definition 5.1.8. Consider maps and a specified homotopy $\alpha : hf \simeq kg$ in a square

$$\begin{array}{ccc} & X & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle h} \\ W & \Downarrow \alpha & Z. \\ {\scriptstyle g}\searrow & & \nearrow{\scriptstyle k} \\ & Y & \end{array}$$

The square, with α, is a homotopy pushout if its comparison map

$$\theta_\alpha : M(f,g) \longrightarrow Z$$

is a homotopy equivalence. This comparison restricts to h, k and sends (w,t) to $\alpha(w,t)$.

Remark 5.1.9. A commutative pushout square need not be a homotopy pushout. For the span $* \xleftarrow{\mathrm{pr}_{S^0}} S^0 \xrightarrow{\mathrm{pr}_{S^0}} *$, where S^0 has two points, the ordinary pushout is a point. Its double mapping cylinder consists of two intervals joining the two target points, hence is a circle. Thus the comparison to the ordinary pushout is not a homotopy equivalence. The specified homotopy in Definition 5.1.8 is part of the gluing data.

5.2 Homology Axioms and Coefficients

We now impose homotopy invariance, additivity, and Mayer–Vietoris. The connecting homomorphisms in the third axiom are additional structure; their existence is not a consequence of functoriality alone. The gluing-first formulation uses homotopy pushouts in place of a separate excision axiom. The original pair-based axioms and uniqueness announcement appear in [ES45, §§4–5, pp. 118–119].

5.2.1 The Homology Axioms and Naturality

Homotopic maps induce the same homomorphism, and disjoint unions correspond to direct sums.

Axiom 5.2.1. Homotopy invariance: for every $n \in \mathbb{Z}$ and every pair of homotopic maps $f_0, f_1 : X \longrightarrow Y$,

$$f_{0,*} = f_{1,*} : A_n(X) \longrightarrow A_n(Y).$$

Axiom 5.2.2. Additivity: for every set-indexed family $(X_\lambda)_{\lambda\in\Lambda}$, the inclusions $\iota_\lambda : X_\lambda \longrightarrow \coprod_{\lambda\in\Lambda} X_\lambda$ induce an isomorphism

$$\bigoplus_{\lambda\in\Lambda} A_n(X_\lambda) \xrightarrow{\cong} A_n\left(\coprod_{\lambda\in\Lambda} X_\lambda\right).$$

The empty family is included, so $A_n(\emptyset) = 0$.

The homotopy in a double mapping cylinder makes $\iota_{X,*}f_* = \iota_{Y,*}g_*$. Consequently the composite of the following two maps is zero:

$$A_n(W) \xrightarrow{(f_*,-g_*)} A_n(X) \oplus A_n(Y) \xrightarrow{\iota_{X,*}+\iota_{Y,*}} A_n(M(f,g)).$$

The gluing axiom asserts exactness and supplies the connecting maps that continue this sequence.

Axiom 5.2.3. Mayer–Vietoris: for every span $X \xleftarrow{f} W \xrightarrow{g} Y$, there are homomorphisms

$$\delta_n : A_n(M(f,g)) \longrightarrow A_{n-1}(W)$$

for which the following five consecutive terms are exact in every degree:

$$A_n(W) \xrightarrow{(f_*,-g_*)} A_n(X) \oplus A_n(Y) \xrightarrow{\iota_{X,*}+\iota_{Y,*}} A_n(M(f,g)) \xrightarrow{\delta_n} A_{n-1}(W) \xrightarrow{(f_*,-g_*)} A_{n-1}(X) \oplus A_{n-1}(Y). \quad (13)$$

The homomorphisms δ_n are natural for maps of spans and their induced maps of double mapping cylinders.

A map of spans consists of maps

$$\alpha : W \longrightarrow W', \qquad \beta : X \longrightarrow X', \qquad \gamma : Y \longrightarrow Y'$$

satisfying $\beta f = f'\alpha$ and $\gamma g = g'\alpha$. It induces

$$\mu : M(f,g) \longrightarrow M(f',g'),$$

whose restrictions are β on X, γ on Y, and $(w,t) \mapsto (\alpha(w),t)$ on the cylinder. Naturality of the connecting maps means that

$$\begin{array}{ccc} A_n(M(f,g)) & \xrightarrow{\delta_n} & A_{n-1}(W) \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \alpha_*} \\ A_n(M(f',g')) & \xrightarrow[\delta'_n]{} & A_{n-1}(W') \end{array}$$

commutes. Exactness in (13) means that the image of each homomorphism is the kernel of the next one.

Definition 5.2.4. A generalized homology theory in this formulation consists of the functors of Definition 5.1.1 and the natural connecting homomorphisms of Axiom 5.2.3, satisfying Axioms 5.2.1–5.2.3. A morphism of theories is a family of natural transformations commuting with the connecting homomorphisms.

Proposition 5.2.5. For a homotopy pushout square as in Definition 5.1.8, there is a long exact sequence with five consecutive terms

$$A_n(W) \xrightarrow{(f_*,-g_*)} A_n(X) \oplus A_n(Y) \xrightarrow{h_*+k_*} A_n(Z) \xrightarrow{\delta_n^Z} A_{n-1}(W) \xrightarrow{(f_*,-g_*)} A_{n-1}(X) \oplus A_{n-1}(Y).$$

Here $\delta_n^Z = \delta_n(\theta_{\alpha,*})^{-1}$.

Proof. Axiom 5.2.1 sends homotopy equivalences to isomorphisms: the image of a homotopy inverse is an inverse. Thus $\theta_{\alpha,*}$ is invertible. Transport (13) through this isomorphism. The identities $\theta_\alpha \iota_X = h$ and $\theta_\alpha \iota_Y = k$ identify its middle map with $h_* + k_*$. Naturality holds for compatible maps of these squares and their chosen cylinder comparisons. □

Remark 5.2.6. An open cover $X = U \cup V$ gives the familiar square

$$\begin{array}{ccc} & U & \\ \nearrow & & \searrow \\ U \cap V & & X. \\ \searrow & & \nearrow \\ & V & \end{array}$$

For a numerable cover, this is a homotopy pushout. Here numerable means that there is a subordinate partition of unity; for two sets, continuous functions $\lambda_U, \lambda_V : X \longrightarrow [0,1]$ have sum 1 and supports contained in U, V, respectively. Let $p : M \to X$ be the comparison from the double mapping cylinder of $U \leftarrow U \cap V \to V$. Define a section $s : X \to M$ by sending $x \in U \cap V$ to $[(x, \lambda_V(x))]$, $x \in U - V$ to its copy in U, and $x \in V - U$ to its copy in V. These formulas agree at height zero and height one under the cylinder identifications. To check continuity, a point outside V has a neighborhood disjoint from $\operatorname{supp}(\lambda_V)$, where the section is the inclusion into U; the analogous statement holds outside U. On $U \cap V$ the formula is continuous. These neighborhoods cover X, proving continuity, and $ps = \mathrm{id}_X$.

On a cylinder point $[(x,t)]$, define the homotopy at time u by the height $(1-u)t + u\lambda_V(x)$. On a point of the U copy use height $u\lambda_V(x)$ when $x \in U \cap V$, and leave it fixed outside V. On the V copy use height $1 - u + u\lambda_V(x)$ on the overlap, and leave it fixed outside U. The same support-neighborhood argument proves continuity on the two copies, and the formulas agree with the cylinder formula at its ends. The quotient universal property therefore gives a continuous homotopy on $M \times I$. Here taking the product of the defining quotient map with I preserves a quotient map as proved in Lecture 7, Lemma 7.1.7. Its endpoints are id_M and sp. Thus p is a homotopy equivalence, and Proposition 5.2.5 applies. For the sphere covers used below, no general partition-of-unity theorem is needed. Write t for the last coordinate on S^m, and take $U = \{t < 1\}$ and $V = \{t > -1\}$. The functions

$$u = \max(0, \tfrac{1}{2} - t), \qquad v = \max(0, t + \tfrac{1}{2}), \qquad \lambda_U = \frac{u}{u+v}, \qquad \lambda_V = \frac{v}{u+v}$$

have positive denominator, sum to one, and have supports contained in U and V, respectively.

5.2.2 Coefficients and the Dimension Axiom

The values on a point remain part of the information in a generalized theory. They determine its values on discrete and contractible spaces, but need not be concentrated in degree zero.

Definition 5.2.7. The coefficient groups of A are the graded abelian groups $G_n = A_n(*)$. The dimension axiom is the additional condition $G_n = 0$ for $n \neq 0$. For ordinary integral homology one further chooses $G_0 \cong \mathbb{Z}$.

Proposition 5.2.8. For every nonempty contractible space X and every discrete set E,

$$A_n(X) \cong G_n, \qquad A_n(E) \cong \bigoplus_{e \in E} G_n.$$

The first isomorphism is induced by the unique map to a point, and the second by the point inclusions.

Proof. The map $\mathrm{pr}_X : X \longrightarrow *$ is a homotopy equivalence, so Axiom 5.2.1 gives the first isomorphism. A discrete set is a disjoint union of points; Axiom 5.2.2 gives the second, including $E = \emptyset$. □

Example 5.2.9. Once ordinary integral homology satisfies the axioms, its shift $A_n(X) = \mathrm{H}_{n-1}(X; \mathbb{Z})$ also satisfies them: shift every degree in its Mayer–Vietoris sequence and connecting maps. Its only nonzero coefficient group is $G_1 = \mathbb{Z}$. This theory violates the dimension axiom while retaining homotopy invariance, additivity, and Mayer–Vietoris. A more general family is $A_n(X) = \bigoplus_{r \in \mathbb{Z}} \mathrm{H}_{n-r}(X; B_r)$ for a fixed graded family of coefficient groups (B_r). Direct sums of exact sequences are exact, so the same argument applies.

Remark 5.2.10. Generalized theories have groups in every integer degree; their long exact sequences do not stop at degree zero. Ordinary singular homology is obtained with its dimension normalization and zero negative-degree groups. For the broader question of uniqueness among theories satisfying the axioms, see [May99, Chapter 15, §2, pp. 119–120].

5.3 Relative Homology and Spheres

The homotopy cofiber defines relative homology. Its Mayer–Vietoris sequence yields the long exact sequence of a pair.

5.3.1 Reduced Homology and Homotopy Cofibers

Write $*$ for a one-point space. Choosing a point of a space splits off its coefficient summand.

Definition 5.3.1. For a space Y with a chosen point y_0, let $\mathrm{pr}_Y : Y \longrightarrow *$ be the unique map and $s : * \longrightarrow Y$ select y_0. Define

$$\widetilde{A}_n(Y) = \ker((\mathrm{pr}_Y)_* : A_n(Y) \longrightarrow G_n).$$

Since $(\mathrm{pr}_Y)_* s_* = \mathrm{id}$, there is an isomorphism $A_n(Y) \cong \widetilde{A}_n(Y) \oplus G_n$. Explicitly, a is sent to $(a - s_*((\mathrm{pr}_Y)_*(a)), (\mathrm{pr}_Y)_*(a))$.

Definition 5.3.2. For a subspace inclusion $\iota_X : X' \hookrightarrow X$, define its homotopy cofiber $C(\iota_X)$ to be the double mapping cylinder of $X \xleftarrow{\iota_X} X' \xrightarrow{\mathrm{pr}_{X'}} *$. Its distinguished point is the copy of $*$. Define

$$A_n(X, X') = \widetilde{A}_n(C(\iota_X)).$$

A map of pairs is a commutative square

$$\begin{array}{ccc} X' & \xhookrightarrow{\iota_X} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xhookrightarrow[\iota_Y]{} & Y. \end{array}$$

It gives a basepoint-preserving map $C(\iota_X) \longrightarrow C(\iota_Y)$, hence a homomorphism of relative groups. Functoriality follows directly from the cylinder construction.

5.3.2 The Long Exact Sequence of a Pair

Mayer–Vietoris for a homotopy cofiber gives the relative exact sequence.

Proposition 5.3.3. Every pair has a natural long exact sequence with five consecutive terms

$$A_n(X') \xrightarrow{\iota_{X,*}} A_n(X) \xrightarrow{q_*} A_n(X, X') \xrightarrow{\delta_n} A_{n-1}(X') \xrightarrow{\iota_{X,*}} A_{n-1}(X).$$

Proof. Let $\kappa : X \longrightarrow C(\iota_X)$ and $s : * \longrightarrow C(\iota_X)$ be the two canonical maps. Mayer–Vietoris gives

$$\cdots \longrightarrow A_n(X') \xrightarrow{(\iota_{X,*}, -(\mathrm{pr}_{X'})_*)} A_n(X) \oplus G_n \xrightarrow{\kappa_* + s_*} A_n(C(\iota_X)) \xrightarrow{\delta_n} A_{n-1}(X') \longrightarrow \cdots,$$

where $\mathrm{pr}_{X'} : X' \longrightarrow *$ is the unique map. Under the splitting of Definition 5.3.1, set $q_*(b) = \kappa_*(b) - s_*((\mathrm{pr}_X)_*(b))$ and define the connecting map δ_n in the claimed sequence by restricting the displayed connecting map to the reduced summand. Change coordinates on the middle direct sum by $(b, g) \mapsto (b, (\mathrm{pr}_X)_* b + g)$. The first map becomes $a \mapsto (\iota_{X,*} a, 0)$; the next becomes $(b, g) \mapsto (q_* b, g)$. The displayed connecting map vanishes on the coefficient summand by exactness. Thus the displayed sequence splits into the claimed sequence and the identity map $G_n \longrightarrow G_n$. Exactness and naturality pass to the claimed summand. □

Definition 5.3.4. For a space X, let $X_+ = X + *$ be the pointed space obtained by adjoining a disjoint basepoint, with basepoint the added point.

Remark 5.3.5. For a cell pair, or a subspace inclusion satisfying the homotopy extension property, the map from the mapping cone to X/X' is a homotopy equivalence (using X_+ when $X' = \emptyset$). Thus the relative groups are the reduced groups of this quotient. For an arbitrary subspace, the homotopy cofiber is the construction to use; the ordinary quotient need not have its homotopy type. The homotopy extension property says that a homotopy on X' extending a map on X extends to a homotopy on X. See May, Chapter 6 and Chapter 14, Section 2.

5.3.3 Suspension and the Homology of Spheres

The homology of spheres is determined by the graded coefficient groups G_n.

Proposition 5.3.6. For every $m \geq 0$ and every $n \in \mathbb{Z}$,

$$\widetilde{A}_n(\mathrm{S}^m) \cong G_{n-m}, \qquad A_n(\mathrm{S}^m) \cong G_n \oplus G_{n-m}.$$

The reduced group is defined using a chosen point of the sphere.

Proof. For $\mathrm{S}^0 = \{x_0, x_1\}$, additivity identifies $A_n(\mathrm{S}^0)$ with $G_n \oplus G_n$. The map to a point is addition, whose kernel consists of $(a, -a)$ and is isomorphic to G_n.

For $m \geq 1$, cover S^m by the complements U, V of its two poles. Both are contractible, and their intersection deformation retracts onto S^{m-1}. The resulting square is a homotopy pushout by Remark 5.2.6. Under the coefficient identifications, its first map is

$$A_n(\mathrm{S}^{m-1}) \longrightarrow G_n \oplus G_n, \qquad a \longmapsto ((\mathrm{pr}_{\mathrm{S}^{m-1}})_*(a), -(\mathrm{pr}_{\mathrm{S}^{m-1}})_*(a)).$$

Its image is the antidiagonal copy of G_n; its kernel is $\widetilde{A}_n(\mathrm{S}^{m-1})$. Exactness gives

$$0 \longrightarrow G_n \longrightarrow A_n(\mathrm{S}^m) \xrightarrow{\delta_n} \widetilde{A}_{n-1}(\mathrm{S}^{m-1}) \longrightarrow 0.$$

The first homomorphism is the map induced by a chosen point in U, so the map $\mathrm{pr}_{\mathrm{S}^m} : \mathrm{S}^m \longrightarrow *$ splits it. Hence δ_n restricts to an isomorphism from $\widetilde{A}_n(\mathrm{S}^m)$ onto $\widetilde{A}_{n-1}(\mathrm{S}^{m-1})$. Induction proves the formula. □

Example 5.3.7. With $G_0 = \mathbb{Z}$ and $G_n = 0$ for $n \neq 0$, Proposition 5.3.6 gives the usual integral homology of spheres. For the shifted theory of Example 5.2.9, the additional sphere summand occurs in degree $m+1$, and the coefficient summand occurs in degree 1. A connected space can therefore have several nonzero groups unrelated to its dimension.

References

[May99] J. P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press, 1999. Further background for Lectures 5 and 7: Chapters 6, 10, and 13–15; Chapter 10, §7, p. 80 treats the double mapping cylinder. https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf

[ES45] S. Eilenberg and N. E. Steenrod, Axiomatic approach to homology theory, *Proceedings of the National Academy of Sciences of the United States of America* **31** (1945), no. 4, 117–120. DOI: 10.1073/pnas.31.4.117.