Homotopy Invariance and First Computations

Algebraic Topology I | Fall 2026 | October 5 and 7

6.1 The Prism Operator and Homotopy Invariance

Singular chains are free abelian on continuous simplices. A triangulation of a simplex times an interval converts a homotopy into a homomorphism of degree one.

6.1.1 Chain Homotopies and the Prism Operator

Use the integer singular chains, their differential, and functoriality from Lecture 4, Definition 4.4.9 and the following paragraphs. We suppress the superscript sing, writing $\mathrm{C}_n(X)$ for the integral singular chains and $\mathrm{H}_n(X;\mathbb{Z})$ for their homology. Write Top for topological spaces and Ab for abelian groups. The affine coface maps of Lecture 4, Definition 4.2.5, are denoted here by $d^i : |\Delta^{n-1}| \to |\Delta^n|$; their vertices remain $e_0, \dots, e_n$. For the classical prism proof of homotopy invariance, see [Hat02, Theorem 2.10 and its proof, pp. 111–113].

Definition 6.1.1. Use chain homotopy from Lecture 1, Definition 1.3.5. For maps $f, g : A_\bullet \to B_\bullet$, a degree-one homotopy K from f to g satisfies $g - f = \partial_B K + K\partial_A$. This endpoint order agrees with the prism construction below.

Proposition 6.1.2. The homology invariance of chain homotopy is Lecture 1, Proposition 1.3.6. In particular, a chain homotopy equivalence induces an isomorphism on homology.

Proof. Apply Lecture 1, Proposition 1.3.6, to the two composites and their homotopies to the identities. □

Write $I = [0,1]$. A homotopy of maps of spaces gives a chain homotopy by triangulating each singular prism.

Theorem 6.1.3. Suppose that

$$f, g : X \longrightarrow Y, \qquad h : X \times I \longrightarrow Y$$

satisfy $h(x,0) = f(x)$ and $h(x,1) = g(x)$. There are homomorphisms

$$P_n : \mathrm{C}_n(X) \longrightarrow \mathrm{C}_{n+1}(Y) \qquad (n \geq 0)$$

with $P_{-1} = 0$ and

$$\partial_{n+1} P_n + P_{n-1}\partial_n = g_{*,n} - f_{*,n} : \mathrm{C}_n(X) \longrightarrow \mathrm{C}_n(Y). \tag{14}$$

Thus singular homology is a homotopy invariant functor in each degree:

$$\mathrm{H}_n(-;\mathbb{Z}) : \mathrm{Top} \longrightarrow \mathrm{Ab}.$$

Proof. The prism operator is constructed in Definition 6.1.5; Proposition 6.1.9 proves (14). For a cycle α, that identity gives

$$g_{*,n}(\alpha) - f_{*,n}(\alpha) = \partial_{n+1} P_n(\alpha).$$

The two images therefore represent the same homology class. □

6.1.2 Cutting the Prism Into Simplices

Fix the base dimension $n \geq 0$. For each prism index $j = 0, \dots, n$, the jth prism simplex has the ordered vertices

$$(e_0, 0), \dots, (e_j, 0), (e_j, 1), \dots, (e_n, 1).$$

Write $s_i = (e_i, 0)$ for the bottom vertices. The affine parametrization is

$$\tau_j : |\Delta^{n+1}| \longrightarrow |\Delta^n| \times I, \qquad \tau_j(e_k) = \begin{cases} (e_k, 0), & 0 \leq k \leq j, \\ (e_{k-1}, 1), & j+1 \leq k \leq n+1. \end{cases}$$

Here k is a vertex position in the prism simplex. The e_k on the left is a standard vertex of the source; those on the right are standard vertices of the base. Affine extension gives the coordinate formula

$$\tau_j : |\Delta^{n+1}| \longrightarrow |\Delta^n| \times I,$$
$$\tau_j(t_0, \dots, t_{n+1}) = \left(\sum_{k=0}^{j} t_k e_k + \sum_{k=j+1}^{n+1} t_k e_{k-1}, \ \sum_{k=j+1}^{n+1} t_k \right).$$

Thus τ_j identifies the two source coordinates in positions $j, j+1$ in the base, and adds the coordinates after position j to obtain the height.

Remark 6.1.4. For a point (x, t) in the prism, now write $x = \sum_{i=0}^{n} t_i e_i$. It belongs to the jth piece exactly when

$$\sum_{i=j+1}^{n} t_i \ \leq t \ \leq \sum_{i=j}^{n} t_i. \tag{15}$$

Indeed, the unique coordinates in the source of τ_j are

$$\left(t_0, \dots, t_{j-1}, \ \sum_{i=j}^{n} t_i - t, \ t - \sum_{i=j+1}^{n} t_i, \ t_{j+1}, \dots, t_n \right).$$

They sum to 1, and they are nonnegative exactly under (15). An empty sum is zero, and an empty string of coordinates is omitted.

For a fixed base point, the endpoints of the intervals in (15) occur in order:

$$0 \leq t_n \leq t_{n-1} + t_n \leq \cdots \leq t_0 + \cdots + t_n = 1.$$

Hence the pieces cover the prism. For $j < j'$, membership in both pieces forces

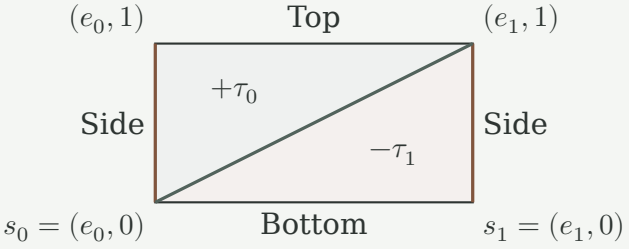
$$t = \sum_{i=j+1}^{n} t_i = \sum_{i=j'}^{n} t_i, \qquad t_{j+1} = \cdots = t_{j'-1} = 0.$$

Their intersection is exactly the common face whose ordered vertices are

$$(e_0, 0), \dots, (e_j, 0), (e_{j'}, 1), \dots, (e_n, 1).$$

Here a face means all linear combinations of the listed vertices with nonnegative coefficients summing to 1. Thus the pieces meet along faces and form a triangulation.

For $n = 1$ the picture is a square divided into two triangles:



The open diagonal is inside the square. The vertical edges lie over the boundary points of the base interval. This distinction persists in every dimension.

Definition 6.1.5. For a singular simplex σ, define σ_j by the composite

$$|\Delta^{n+1}| \xrightarrow{\tau_j} |\Delta^n| \times I \xrightarrow{\sigma \times \mathrm{id}_I} X \times I \xrightarrow{h} Y.$$

Explicitly, the middle map is

$$\sigma \times \mathrm{id}_I : |\Delta^n| \times I \longrightarrow X \times I, \qquad (x,t) \longmapsto (\sigma(x), t).$$

The free abelian universal property extends the following assignment uniquely to a homomorphism:

$$P_n : \mathrm{C}_n(X) \longrightarrow \mathrm{C}_{n+1}(Y), \qquad \sigma \longmapsto \sum_{j=0}^{n} (-1)^j \sigma_j.$$

6.1.3 Prism Faces and Their Cancellation

The boundary of the signed prism consists of its two ends and the prisms on its base faces. This identity gives homotopy invariance and the first absolute computations.

The terms in the boundary are indexed by a Cartesian product:

$$(j,k) \in \{0,\dots,n\} \times \{0,\dots,n+1\}, \qquad \partial_{n+1} P_n(\sigma) = \sum_{j=0}^{n} \sum_{k=0}^{n+1} (-1)^{j+k} \sigma_j d^k.$$

The first index chooses the prism piece; the second deletes its vertex in position k. Each corresponding affine face has type

$$\tau_j d^k : |\Delta^n| \longrightarrow |\Delta^n| \times I.$$

The base vertex e_j appears twice in the piece, in positions j and $j+1$. Every other base vertex appears once.

Remark 6.1.6. For $\epsilon \in \{0,1\}$ the endpoint inclusion of the base is

$$\iota_\epsilon : |\Delta^n| \longrightarrow |\Delta^n| \times I, \qquad x \longmapsto (x, \epsilon).$$

The equalities of affine maps, including their signs, are

$$\tau_0 d^0 = \iota_1 : |\Delta^n| \longrightarrow |\Delta^n| \times I, \qquad (-1)^{0+0} = +1,$$

$$\tau_n d^{n+1} = \iota_0 : |\Delta^n| \longrightarrow |\Delta^n| \times I, \qquad (-1)^{n+n+1} = -1.$$

After composition with the homotopy, these give $g\sigma - f\sigma$.

Remark 6.1.7. For $0 \le j < n$, the two faces

$$\tau_j d^{j+1} = \tau_{j+1} d^{j+1} : |\Delta^n| \longrightarrow |\Delta^n| \times I$$

have the same ordered vertices:

$$(e_0,0), \dots, (e_j,0), (e_{j+1},1), \dots, (e_n,1).$$

They are equal as parametrized simplices, not merely as subsets. The equality is the commutative square

$$\begin{array}{ccc} |\Delta^n| & \xrightarrow{d^{j+1}} & |\Delta^{n+1}| \\ {\scriptstyle d^{j+1}}\downarrow & & \downarrow{\scriptstyle \tau_j} \\ |\Delta^{n+1}| & \xrightarrow[\tau_{j+1}]{} & |\Delta^n| \times I. \end{array}$$

Its relative interior lies inside the prism: all base coordinates are positive and both heights occur. Its boundary may meet the outer boundary.

The coefficients of this same ordered face in the two boundaries are

$$(-1)^j(-1)^{j+1} = -1, \qquad (-1)^{j+1}(-1)^{j+1} = +1.$$

Thus the signed prism simplices induce opposite orientations on the face. Their contributions cancel, including after composition with h.

Remark 6.1.8. Deleting the unique vertex over e_i, with $i \neq j$, gives a side face over $t_i = 0$. Its position is i for $i < j$ and $i+1$ for $i > j$. In the lower-dimensional prism the symbols τ_{j-1} and τ_j use the same affine definition, now with source $|\Delta^n|$ and target $|\Delta^{n-1}| \times I$. The two cases give the squares

$$\begin{array}{ccc} |\Delta^n| & \xrightarrow{d^i} & |\Delta^{n+1}| \\ {\scriptstyle \tau_{j-1}}\downarrow & & \downarrow{\scriptstyle \tau_j} \\ |\Delta^{n-1}| \times I & \xrightarrow[d^i \times \mathrm{id}_I]{} & |\Delta^n| \times I \end{array} \qquad (i < j),$$

$$\begin{array}{ccc} |\Delta^n| & \xrightarrow{d^{i+1}} & |\Delta^{n+1}| \\ {\scriptstyle \tau_j}\downarrow & & \downarrow{\scriptstyle \tau_j} \\ |\Delta^{n-1}| \times I & \xrightarrow[d^i \times \mathrm{id}_I]{} & |\Delta^n| \times I \end{array} \qquad (i > j).$$

Both squares commute by equality of ordered vertices. Omitting e_i changes the repeated vertex index from j to $j-1$ when $i < j$ and leaves it unchanged when $i > j$.

6.1.4 The Prism Boundary Identity

Proposition 6.1.9. For the prism operator of Definition 6.1.5 and every $n \geq 0$, with $P_{-1} = 0$, one has

$$\partial_{n+1}P_n + P_{n-1}\partial_n = g_{*,n} - f_{*,n} : \mathrm{C}_n(X) \longrightarrow \mathrm{C}_n(Y).$$

Proof. Fix a singular n-simplex σ with $n \geq 1$. Expand the two compositions:

$$\partial_{n+1}P_n(\sigma) = \sum_{j=0}^{n}\sum_{k=0}^{n+1}(-1)^{j+k}\sigma_j d^k \qquad \text{Definition 6.1.5 and the boundary formula.}$$

$$P_{n-1}\partial_n(\sigma) = \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}(\sigma d^i)_j \qquad i \text{ selects the base face; } j \text{ its prism piece.}$$

Here $(\sigma d^i)_j$ denotes the jth prism simplex of the singular face σd^i. The endpoint terms in the first expansion are

$$+\sigma_0 d^0 = h(\sigma \times \mathrm{id}_I)\iota_1 = g\sigma \qquad (j,k) = (0,0); \text{ Remark 6.1.6.}$$
$$-\sigma_n d^{n+1} = -h(\sigma \times \mathrm{id}_I)\iota_0 = -f\sigma \qquad (j,k) = (n, n+1); \text{ Remark 6.1.6.}$$

The remaining vertices have height 1 and 0, respectively. These are the two full base simplices; the second expansion consists of prisms on base faces.

The table matches every other term. The ordered pairs refer to the prism-piece index and deleted position in ∂P.

Indices in ∂P	Signed term	Opposite signed term
$(j, j+1),\ 0 \le j < n$	$-\sigma_j d^{j+1}$	$+\sigma_{j+1} d^{j+1}$ in ∂P
$(j+1, i),\ 0 \le i \le j < n$	$(-1)^{i+j+1}\sigma_{j+1} d^{i}$	$(-1)^{i+j}(\sigma d^{i})_{j}$ in $P\partial$
$(j, i+1),\ 0 \le j < i \le n$	$(-1)^{i+j+1}\sigma_{j} d^{i+1}$	$(-1)^{i+j}(\sigma d^{i})_{j}$ in $P\partial$

In the first row the paired index is $(j+1, j+1)$, and equality of the two faces is Remark 6.1.7. The last two rows use the commuting squares in Remark 6.1.8; their index ranges partition all (i, j) with $0 \le i \le n$ and $0 \le j < n$. Thus the $(n+1)(n+2)$ terms of ∂P consist of 2 endpoints, $2n$ interior terms, and $n(n+1)$ side terms. The interior terms cancel in pairs; the side terms match all $n(n+1)$ terms of $P\partial$ with opposite signs. Explicitly,

$$
\begin{aligned}
\partial_{n+1} P_n(\sigma) &= g\sigma - f\sigma && \text{endpoint indices } (0,0),(n,n+1).\\
&\quad + \sum_{j=0}^{n-1} \left(-\sigma_j d^{j+1} + \sigma_{j+1} d^{j+1}\right) && \text{interior pairs, first row.}\\
&\quad - \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} (\sigma d^{i})_{j} && \text{side indices } (j+1, i), \text{ second row.}\\
&\quad - \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} (\sigma d^{i})_{j} && \text{side indices } (j, i+1), \text{ third row.}\\
&= g\sigma - f\sigma - P_{n-1}\partial_n(\sigma) && \text{cancel interior pairs and join the side ranges.}
\end{aligned}
$$

For $n = 0$, the prism is one interval and $\partial_1 P_0(\sigma) = g\sigma - f\sigma$, with $P_{-1} = 0$. Adding $P_{n-1}\partial_n$ and extending from generators proves (14) in every degree. □

By the general homotopy-invariance proposition in Lecture 5, homotopy equivalences induce homology isomorphisms.

6.2 Absolute and Relative Homology

6.2.1 Components, Points, and Disjoint Unions

Write $\pi_0^{\mathrm{path}}(X)$ for the set of path components, and $\mathbb{Z}[E]$ for the free abelian group on a set E.

Proposition 6.2.1. There is a natural isomorphism

$$\eta_X : \mathrm{H}_0(X;\mathbb{Z}) \longrightarrow \mathbb{Z}[\pi_0^{\mathrm{path}}(X)], \qquad [x] \longmapsto \text{the generator for the component of } x.$$

Proof. Let B be an abelian group. A homomorphism

$$\alpha : \mathrm{C}_0(X) \longrightarrow B$$

is an assignment of an element of B to each point. The boundary of a path is its endpoint minus its starting point. Thus $\alpha\partial_1 = 0$ exactly when the assignment is constant on path components.

The cokernel property gives the unique factorization

$$\begin{array}{ccccc} \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{q} & \mathrm{H}_0(X;\mathbb{Z}) \\ & & & \searrow^{\alpha} & \downarrow{\scriptstyle \overline{\alpha}} \\ & & & & B. \end{array}$$

This is the free abelian universal property for the set of path components. It proves the isomorphism, and its description on points proves naturality. □

Proposition 6.2.2. Use the homology of a point from Lecture 4, Example 4.4.8: the coefficient group is $\mathbb{Z}$ in degree zero and zero otherwise.

Proof. The singular simplicial set of a point is Δ^0. Apply the constant simplicial group computation of Lecture 4, Example 4.1.7, as specialized in Lecture 4, Example 4.4.8. □

Proposition 6.2.3. For a family $(X_\lambda)_{\lambda\in\Lambda}$, the summand inclusions induce isomorphisms

$$\bigoplus_{\lambda\in\Lambda} \mathrm{C}_\bullet(X_\lambda) \xrightarrow{\cong} \mathrm{C}_\bullet\left(\coprod_{\lambda\in\Lambda} X_\lambda\right), \qquad \bigoplus_{\lambda\in\Lambda} \mathrm{H}_n(X_\lambda;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_n\left(\coprod_{\lambda\in\Lambda} X_\lambda;\mathbb{Z}\right).$$

Proof. Each singular simplex has connected nonempty domain, so its image lies in one summand of the disjoint union. The generators therefore split by summand, and every face stays in the same summand. This proves the chain isomorphism. In a direct sum every element has finite support, so both cycles and boundaries are direct sums of the cycles and boundaries in the summands. Taking their quotient proves the homology isomorphism. □

The same argument splits singular chains over path components, even when those components are not open: a simplex has path connected domain and hence lies in one path component, with its subspace topology. In particular, for a discrete space E,

$$\mathrm{H}_0(E;\mathbb{Z}) = \mathbb{Z}[E], \qquad \mathrm{H}_n(E;\mathbb{Z}) = 0 \quad (n>0).$$

For the empty space all chain and homology groups are zero.

6.2.2 Contractible Spaces and Deformation Retractions

Definition 6.2.4. A space is *contractible* if it is homotopy equivalent to the one-point space.

Remark 6.2.5. Equivalently, it is nonempty and its identity is homotopic to a constant map: the map to the point and the inclusion of that constant value are homotopy inverses.

Proposition 6.2.6. For a contractible space X,

$$\mathrm{H}_0(X;\mathbb{Z}) \cong \mathbb{Z}, \qquad \mathrm{H}_n(X;\mathbb{Z}) = 0 \quad (n>0).$$

Proof. Choose a point x_0 to which X contracts. The maps

$$\mathrm{pr}_X : X \longrightarrow *, \qquad g : * \longrightarrow X, \qquad * \longmapsto x_0$$

satisfy $\mathrm{pr}_X\, g = \mathrm{id}_*$ and $g\,\mathrm{pr}_X \simeq \mathrm{id}_X$. Their induced maps

$$(\mathrm{pr}_X)_* : \mathrm{H}_n(X;\mathbb{Z}) \longrightarrow \mathrm{H}_n(*;\mathbb{Z}), \qquad g_* : \mathrm{H}_n(*;\mathbb{Z}) \longrightarrow \mathrm{H}_n(X;\mathbb{Z})$$

are therefore inverse isomorphisms. Apply the computation for a point. □

For a nonempty convex subset $V \subseteq \mathbb{R}^m$ and $x_0 \in V$, the contraction is

$$h : V \times I \longrightarrow V, \qquad (x,t) \longmapsto (1-t)x + tx_0.$$

This applies to Euclidean spaces, balls, intervals, and standard simplices. It also applies when V is star-shaped about x_0, meaning that it contains the line segment joining x_0 to each of its points.

Definition 6.2.7. A *deformation retraction* consists of an inclusion and a retraction

$$\iota : X' \hookrightarrow X, \qquad r : X \longrightarrow X'$$

and a homotopy $h : X \times I \longrightarrow X$ satisfying

$$r\iota = \mathrm{id}_{X'}, \qquad h(x,0) = x, \qquad h(x,1) = \iota r(x), \qquad h(\iota(x'),t) = \iota(x').$$

Then ι and r are homotopy inverse maps, so

$$\iota_* : \mathrm{H}_n(X';\mathbb{Z}) \longrightarrow \mathrm{H}_n(X;\mathbb{Z}), \qquad r_* : \mathrm{H}_n(X;\mathbb{Z}) \longrightarrow \mathrm{H}_n(X';\mathbb{Z})$$

are inverse isomorphisms. For example,

$$r : \mathbb{R} - \{0\} \longrightarrow \{-1,1\}, \qquad x \longmapsto x/|x|,$$

$$h : (\mathbb{R} - \{0\}) \times I \longrightarrow \mathbb{R} - \{0\}, \qquad (x,t) \longmapsto (1-t)x + t\,x/|x|$$

give a deformation retraction. The interpolation remains in the half-line containing x. Thus the homology is $\mathbb{Z}^2$ in degree zero and zero in positive degrees. Removing m distinct points from $\mathbb{R}$ similarly leaves $m+1$ contractible components.

6.2.3 Pairs, Quotients, and the Exact Sequence

Quotient chain complexes give the relative exact sequence. Homotopies preserving the subspaces induce chain homotopies on these quotients.

Definition 6.2.8. A pair is a subspace inclusion

$$\iota : X' \hookrightarrow X.$$

Write (X, X') for this pair. The induced inclusion identifies the singular chains of X' with a subcomplex. Define

$$\mathrm{C}_\bullet(X,X') = \mathrm{C}_\bullet(X)/\mathrm{C}_\bullet(X'), \qquad \mathrm{H}_n(X,X';\mathbb{Z}) = \mathrm{H}_n(\mathrm{C}_\bullet(X,X')).$$

A map of pairs is a commutative square

$$\begin{array}{ccc} X' & \xhookrightarrow{\iota_X} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xhookrightarrow[\iota_Y]{} & Y. \end{array}$$

The relative chain complex is a cokernel. For any chain complex $B_\bullet$ and chain map

$$\alpha : \mathrm{C}_\bullet(X) \longrightarrow B_\bullet, \qquad \alpha\iota_* = 0,$$

there is a unique induced map

$$\overline{\alpha} : \mathrm{C}_\bullet(X,X') \longrightarrow B_\bullet$$

making the right square commute; the left square is a pushout:

$$\begin{array}{ccccc} \mathrm{C}_\bullet(X') & \xrightarrow{\iota_*} & \mathrm{C}_\bullet(X) & \xrightarrow{\alpha} & B_\bullet \\ \downarrow & & \downarrow{\scriptstyle q} & & \| \\ 0 & \longrightarrow & \mathrm{C}_\bullet(X,X') & \xrightarrow[\overline{\alpha}]{} & B_\bullet. \end{array}$$

For a map of pairs this constructs the relative chain map in the square

$$\begin{array}{ccc} \mathrm{C}_\bullet(X) & \xrightarrow{q_X} & \mathrm{C}_\bullet(X,X') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{C}_\bullet(Y) & \xrightarrow[q_Y]{} & \mathrm{C}_\bullet(Y,Y'). \end{array}$$

Uniqueness gives compatibility with identities and composition, hence functoriality.

A relative cycle is represented by $\alpha \in \mathrm{C}_n(X)$ with $\partial_n \alpha \in \mathrm{C}_{n-1}(X')$. Two such representatives have the same relative homology class exactly when their difference is a boundary in X plus a chain in X'.

Theorem 6.2.9. Every pair has a natural long exact sequence with five consecutive terms

$$\mathrm{H}_n(X';\mathbb{Z}) \xrightarrow{\iota_*} \mathrm{H}_n(X;\mathbb{Z}) \xrightarrow{q_*} \mathrm{H}_n(X,X';\mathbb{Z}) \xrightarrow{\delta_n} \mathrm{H}_{n-1}(X';\mathbb{Z}) \xrightarrow{\iota_*} \mathrm{H}_{n-1}(X;\mathbb{Z})$$

ending in

$$\mathrm{H}_0(X';\mathbb{Z}) \xrightarrow{\iota_*} \mathrm{H}_0(X;\mathbb{Z}) \xrightarrow{q_*} \mathrm{H}_0(X,X';\mathbb{Z}) \longrightarrow 0.$$

The connecting homomorphism is

$$\delta_n : \mathrm{H}_n(X,X';\mathbb{Z}) \longrightarrow \mathrm{H}_{n-1}(X';\mathbb{Z}), \qquad [\alpha] \longmapsto [\partial_n \alpha].$$

Proof. Apply the long exact sequence of chain complexes from Lecture 2 to

$$0 \longrightarrow \mathrm{C}_\bullet(X') \xrightarrow{\iota_*} \mathrm{C}_\bullet(X) \xrightarrow{q} \mathrm{C}_\bullet(X,X') \longrightarrow 0.$$

For a relative cycle, $\partial_n \alpha$ lies in the subcomplex and is a cycle because $\partial_{n-1}\partial_n = 0$. Replacing α by $\alpha + \partial_{n+1}\beta + \alpha'$ with $\alpha' \in \mathrm{C}_n(X')$ changes its boundary by $\partial_n \alpha'$, so the connecting class is well defined. The chain-complex theorem supplies exactness and naturality. □

Example 6.2.10. Take $X = I$ and $X' = \partial I = \{0,1\}$. Contractibility of I and the computation for discrete spaces reduce the exact sequence to

$$0 \longrightarrow \mathrm{H}_1(I,\partial I;\mathbb{Z}) \xrightarrow{\delta_1} \mathbb{Z}^2 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow \mathrm{H}_0(I,\partial I;\mathbb{Z}) \longrightarrow 0,$$

where

$$\varepsilon : \mathbb{Z}^2 \longrightarrow \mathbb{Z}, \qquad (a,b) \longmapsto a+b.$$

Both endpoint generators map to the component generator of I. The map is surjective, and its kernel is $\mathbb{Z}(-1,1)$. Thus

$$\mathrm{H}_n(I,\partial I;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 1, \\ 0, & n \neq 1. \end{cases}$$

The singular edge

$$\sigma : |\Delta^1| \longrightarrow I, \qquad (t_0,t_1) \longmapsto t_1$$

has boundary equal to the point generator at 1 minus the point generator at 0. Thus $\delta_1[\sigma] = (-1, 1)$ and $[\sigma]$ is a generator.

Reflection is the map of pairs induced by

$$\rho : I \longrightarrow I, \qquad t \longmapsto 1 - t.$$

Its restriction to the endpoints is

$$\rho' : \partial I \longrightarrow \partial I, \qquad t \longmapsto 1 - t.$$

Naturality gives

$$\begin{array}{ccc} \mathrm{H}_1(I, \partial I; \mathbb{Z}) & \xrightarrow{\delta_1} & \mathrm{H}_0(\partial I; \mathbb{Z}) \\ \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho'_*} \\ \mathrm{H}_1(I, \partial I; \mathbb{Z}) & \xrightarrow[\delta_1]{} & \mathrm{H}_0(\partial I; \mathbb{Z}). \end{array}$$

The right map exchanges the endpoint generators, sending $(-1, 1)$ to its negative. Injectivity of δ_1 therefore gives $\rho_*([\sigma]) = -[\sigma]$.

6.2.4 Relative Homotopies and Reduced Homology

A homotopy preserving the subspaces descends to the quotient chain complexes.

Proposition 6.2.11. Suppose that

$$f, g : (X, X') \longrightarrow (Y, Y')$$

are joined by a homotopy

$$h : X \times I \longrightarrow Y, \qquad h(X' \times I) \subseteq Y'.$$

Then the induced maps agree:

$$f_* = g_* : \mathrm{H}_n(X, X'; \mathbb{Z}) \longrightarrow \mathrm{H}_n(Y, Y'; \mathbb{Z}).$$

Proof. For a simplex in X', all its prism simplices have image in Y'. Hence the prism homomorphisms restrict and descend:

$$P_n : \mathrm{C}_n(X') \longrightarrow \mathrm{C}_{n+1}(Y'), \qquad \overline{P}_n : \mathrm{C}_n(X, X') \longrightarrow \mathrm{C}_{n+1}(Y, Y').$$

Identity (14) descends to the quotient. It makes the relative chain maps chain homotopic, so they induce equal homology maps. □

The subspace condition is essential. The contraction

$$h : I \times I \longrightarrow I, \qquad (x, t) \longmapsto (1 - t)x$$

moves the endpoint 1 into the interior for $0 < t < 1$. It is therefore not a homotopy of the pair $(I, \partial I)$.

Definition 6.2.12. For a nonempty space X, the *augmentation* is the chain map

$$\varepsilon : \mathrm{C}_\bullet(X) \longrightarrow \mathbb{Z}[0],$$

whose degree-zero component is

$$\varepsilon_0 : \mathrm{C}_0(X) \longrightarrow \mathbb{Z}, \qquad \sum_i a_i x_i \longmapsto \sum_i a_i,$$

and whose positive-degree components are zero. Here $\mathbb{Z}[0]$ is concentrated in degree zero. Every path boundary has coefficient sum zero, so this is a chain map. It is surjective because X is nonempty.

Definition 6.2.13. The *reduced chain complex* of a nonempty space X is the kernel of its augmentation, equivalently the pullback:

$$\begin{array}{ccc} \widetilde{\mathrm{C}}_\bullet(X) & \longrightarrow & \mathrm{C}_\bullet(X) \\ \downarrow & & \downarrow{\scriptstyle \varepsilon} \\ 0 & \longrightarrow & \mathbb{Z}[0]. \end{array}$$

Its homology is the *reduced homology* $\widetilde{\mathrm{H}}_n(X;\mathbb{Z})$.

The kernel exact sequence gives

$$\widetilde{\mathrm{H}}_n(X;\mathbb{Z}) \cong \mathrm{H}_n(X;\mathbb{Z}) \quad (n>0), \qquad \widetilde{\mathrm{H}}_0(X;\mathbb{Z}) = \ker(\varepsilon_* : \mathrm{H}_0(X;\mathbb{Z}) \longrightarrow \mathbb{Z}).$$

The prism homomorphisms restrict to the reduced complexes, so reduced homology is homotopy invariant.

If X is contractible and X' is nonempty, the exact sequence of the pair gives isomorphisms

$$\delta_n : \mathrm{H}_n(X,X';\mathbb{Z}) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{n-1}(X';\mathbb{Z}) \quad (n\geq 1), \qquad \mathrm{H}_0(X,X';\mathbb{Z}) = 0.$$

For $n \geq 2$ both adjacent positive-degree groups of X vanish. For $n = 1$ the image of δ_1 is the kernel of the augmentation on $\mathrm{H}_0(X';\mathbb{Z})$.

For example, let X' be a finite discrete subspace consisting of $m \geq 1$ points in a contractible space X. Then

$$\mathrm{H}_1(X,X';\mathbb{Z}) \cong \ker(\varepsilon : \mathbb{Z}^m \longrightarrow \mathbb{Z}) \cong \mathbb{Z}^{m-1},$$

where ε sums the coordinates; all other relative groups vanish. Choose one marked point and paths from it to the others. Their relative classes form a basis because their boundaries are a basis of the augmentation kernel. In particular, three marked points in an interval or a disk give relative first homology $\mathbb{Z}^2$.

For an open cover, the small-chain subdivision theorem identifies the homology of chains supported in individual cover members with singular homology. The resulting short exact sequence of chain complexes induces the Mayer–Vietoris sequence.

References

[Hat02] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002. See also §2.2, pp. 149–150 for mapping tori.