Homotopy Pushouts and Mayer–Vietoris

Algebraic Topology I | Fall 2026 | October 12 and 14

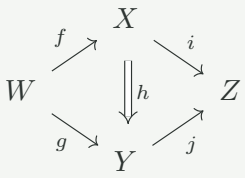
7.1 Homotopy Pushouts and Their Chain Models

7.1.1 Double Mapping Cylinders and Their Universal Property

Write $I = [0,1]$. For a space W, write $\mathrm{pr}_W : W \to *$ for its unique map to a point. We use Top for the category of topological spaces and continuous maps. As in Lecture 5, homotopy pushout means that the comparison is a homotopy equivalence.

Definition 7.1.1. Use the double mapping cylinder $\mathrm{Cyl}(f,g)$ of Lecture 5, Definition 5.1.7. Write ι_X, ι_Y for its canonical maps and $\eta(w,t) = [w,t]$ for its canonical homotopy from $\iota_X f$ to $\iota_Y g$.

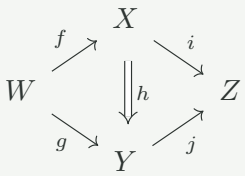
Proposition 7.1.2. Use the quotient universal property following Lecture 5, Definition 5.1.7. In the present notation, a map $\mathrm{Cyl}(f,g) \to Z$ is specified by the diagram



naturally in Z.

Proof. The quotient argument is given after Lecture 5, Definition 5.1.7; postcomposition gives naturality in Z. □

Definition 7.1.3. Use homotopy pushout squares as in Lecture 5, Definition 5.1.8, in the form



The square is equipped with a homotopy h from if to jg. Its comparison map is $\theta_h : \mathrm{Cyl}(f,g) \to Z$.

The specified homotopy is part of this definition. Even when $if = jg$, a nonconstant chosen homotopy can change the comparison map. For a strictly commutative square, the constant homotopy is the usual choice.

A map out of the ordinary pushout $X \underset{W}{+} Y$ requires equality of the two composites. A map out of $\mathrm{Cyl}(f,g)$ requires a chosen homotopy. Passing to homotopy classes therefore does not turn this construction into an ordinary pushout in the homotopy category: the homotopy between the composites remains part of the mapping data.

Example 7.1.4. Take $X = Y = *$, where $*$ is a one-point space. For the two maps pr_W, the double mapping cylinder $\mathrm{Cyl}(\mathrm{pr}_W, \mathrm{pr}_W)$ is the *unreduced suspension*

$$\mathrm{S}(W) = ((W \times I) + \{*_0\} + \{*_1\})/((w,0) \sim *_0,\ (w,1) \sim *_1).$$

Here $*_0$ and $*_1$ are distinct vertices. In particular, $\mathrm{S}(\mathrm{S}^0)$ is a circle: its two intervals have the same two endpoints. The ordinary pushout of $* \xleftarrow{\mathrm{pr}_{\mathrm{S}^0}} \mathrm{S}^0 \xrightarrow{\mathrm{pr}_{\mathrm{S}^0}} *$ is a point. Thus an ordinary pushout need not be a homotopy pushout. More generally, $\mathrm{S}(\mathrm{S}^m) \cong \mathrm{S}^{m+1}$ for $m \geq 0$.

7.1.2 Cofibrations and Geometric Gluing

A Hurewicz cofibration permits a homotopy on a subspace to extend to the whole space. This extension makes ordinary gluing agree with homotopy gluing. The comparison below is the double-mapping-cylinder lemma of May, Chapter 10, §7, p. 80; we give the extension argument explicitly.

Definition 7.1.5. A continuous map $f : X \longrightarrow Y$ is a *Hurewicz cofibration* if it has the homotopy extension property: for every space Z, every map $h : Y \longrightarrow Z$, and every homotopy $\alpha : X \times I \longrightarrow Z$ starting at hf, there is a homotopy $\beta : Y \times I \longrightarrow Z$ starting at h satisfying $\beta(f(x), t) = \alpha(x, t)$ for all $(x, t) \in X \times I$.

For a subspace inclusion, equivalently, the homotopy extension problem is the extension in the diagram

$$\begin{array}{ccc} (Y \times \{0\}) \underset{X \times \{0\}}{\cup} (X \times I) & \longrightarrow & Z \\ \downarrow & \nearrow_{\beta} & \\ Y \times I & & \end{array}$$

for every continuous map on the upper-left subspace. We prove the disk and finite-cell cases used below; see also May, Chapter 6, §1, and Chapter 10, §1.

Regard D^n as the unit disk in $\mathbb{R}^n$ and put $\mathrm{S}^{-1} = \emptyset$. For $(x, t) \in \mathrm{D}^n \times I$, define

$$m(x,t) = \max\{\|x\|, 1 - t/2\},$$
$$\rho(x,t) = \left(\frac{x}{m(x,t)},\ 2 + \frac{t-2}{m(x,t)} \right).$$

The denominator lies in $[1/2, 1]$, the first coordinate has norm at most one, and the second lies in $[0, t]$. According to which term attains the maximum, the image lies on the side or the bottom of the cylinder. On either of those subspaces $m = 1$, so ρ is a continuous retraction onto

$$(\mathrm{D}^n \times \{0\}) \cup (\mathrm{S}^{n-1} \times I).$$

For $n = 0$, it retracts the interval onto its bottom endpoint. A map on the disk and a homotopy on its boundary paste continuously on this union when they agree at time zero. Composing with ρ proves the homotopy extension property for $\mathrm{S}^{n-1} \hookrightarrow \mathrm{D}^n$.

If a disk is attached to Y' through $f : \mathrm{S}^{n-1} \to Y'$, pull the given homotopy on Y' back along f, extend it over the disk by this construction, and glue. Continuity follows because taking the product of the attaching pushout with I preserves the pushout, by Lemma 7.1.7 below. Repetition proves the extension property for any finite sequence of disk attachments. In particular, a subcomplex inclusion $X' \hookrightarrow X$ in a finite cell complex has this property: attach the remaining cells in increasing dimension, so that their boundaries are already present.

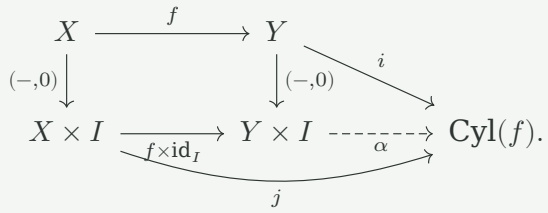
Proposition 7.1.6. Let $f : X \hookrightarrow Y$ be a Hurewicz cofibration and $g : X \longrightarrow Z$ any continuous map. Then the quotient map

$$q : \mathrm{Cyl}(f, g) \longrightarrow Y \underset{X}{+} Z$$

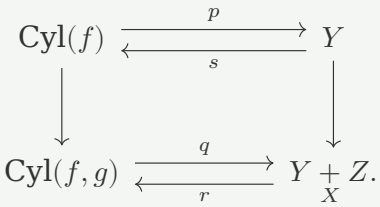
given on the cylinder by $[x, t] \mapsto [f(x)] = [g(x)]$ is a homotopy equivalence. Here $[x, t]$ denotes the image of $(x, t) \in X \times I$ under the cylinder structure map to the double mapping cylin-

der. Consequently, the ordinary pushout square, with its constant homotopy, is a homotopy pushout.

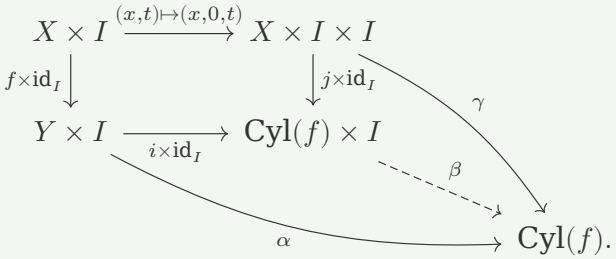
Proof. Let $i : Y \to \mathrm{Cyl}(f)$ and $j : X \times I \to \mathrm{Cyl}(f)$ be the canonical maps, and put $k(x) = j(x, 1)$. The collapse $p : \mathrm{Cyl}(f) \to Y$ satisfies $pi = \mathrm{id}_Y$ and $pj(x, u) = f(x)$. The homotopy extension property supplies the dashed extension:



Set $s(y) = \alpha(y, 1)$. Since $sf = k$ and $pk = f$, the maps s, p glue to the identity on Z, giving the lower maps in the two commuting squares below



The homotopy $p\alpha$ joins id_Y to ps relative to $f(X)$. To construct the corresponding relative homotopy on $\mathrm{Cyl}(f)$, glue α to $\gamma(x, u, t) = j(x, u + t(1 - u))$ through the cylinder pushout:



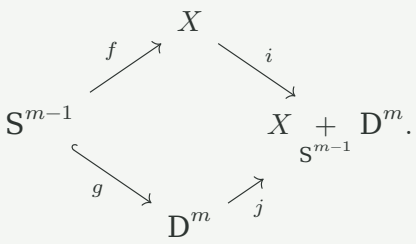
The restrictions agree because $\alpha(f(x), t) = j(x, t) = \gamma(x, 0, t)$. Thus β joins $\mathrm{id}_{\mathrm{Cyl}(f)}$ to sp relative to $k(X)$: at $u = 1$, the map γ is constantly $k(x)$. Gluing $p\alpha$ and β to the constant homotopy on Z gives $\mathrm{id} \simeq qr$ and $\mathrm{id} \simeq rq$.

These gluings are continuous by Lemma 7.1.7. □

Lemma 7.1.7. If $q : X \to Y$ is a quotient map, then $q \times \mathrm{id}_I : X \times I \to Y \times I$ is a quotient map. Consequently, taking the product with I preserves pushouts.

Proof. Suppose $U \subseteq Y \times I$ has open inverse image. For $(y, t) \in U$, its slice U_y is open: choose $x \in q^{-1}(y)$ and restrict that inverse image to $\{x\} \times I$. Choose a compact interval $K \subseteq U_y$ with $t \in \mathrm{int}_I K$, and put $V = \{y' \in Y : \{y'\} \times K \subseteq U\}$. For each $x \in q^{-1}(V)$, compactness of K gives a neighborhood of x whose product with K lies in the inverse image of U. Thus $q^{-1}(V)$ is open, so V is open by the quotient property. The neighborhood $V \times \mathrm{int}_I K$ of (y, t) lies in U, proving the first assertion. A pushout is the quotient of a disjoint union by the gluing identifications; products with I preserve that quotient and distribute over the disjoint union. □

Example 7.1.8. For a map $f : \mathrm{S}^{m-1} \longrightarrow X$, attaching an m-cell is the pushout



The inclusion $g : \mathrm{S}^{m-1} \hookrightarrow \mathrm{D}^m$ is a Hurewicz cofibration. Proposition 7.1.6, with the two legs interchanged, makes this a homotopy pushout. The same conclusion holds for gluing finite cell complexes along a common subcomplex. Taking both legs to be $\mathrm{S}^{m-1} \hookrightarrow \mathrm{D}^m$ gives S^m, obtained by gluing two disks along their boundary.

Example 7.1.9. For $f : W \longrightarrow X$, the space $\mathrm{Cyl}(f, \mathrm{pr}_W)$ is the *mapping cone*

$$\mathrm{Cone}(f) = (X + (W \times I) + \{*\})/((w,0) \sim f(w),\ (w,1) \sim *).$$

Thus the square $W \longrightarrow X$, $* \longrightarrow \mathrm{Cone}(f)$, with its cylinder homotopy, is a homotopy pushout. If W is nonempty and f is a Hurewicz cofibration, Proposition 7.1.6 gives $\mathrm{Cone}(f) \simeq X/W$, where X/W is defined by $w \sim *$ for all $w \in W$. For the identity map of W, this cone is contractible. When $W = \emptyset$, the mapping cone is X_+.

7.1.3 Subdivision

Singular chains preserve homotopy pushouts. We prove this first over $\mathbb{Z}$, using small chains, and then tensor with arbitrary coefficients. Mayer–Vietoris and excision follow from this preservation theorem.

For a space Z, write $\mathrm{C}_\bullet(Z)$ for its singular chain complex with integer coefficients. For an open cover $Z = U \cup V$, put $W = U \cap V$ and let $\mathrm{C}_\bullet^{\{U,V\}}(Z)$ denote the subcomplex of $\mathrm{C}_\bullet(Z)$ generated by singular simplices whose images lie in U or V. Faces of such simplices remain in the same set, so this is a subcomplex.

The small-chains theorem below makes the passage from an open cover to singular homology possible. We prove both the subdivision construction and the chain homotopy equivalence; the same construction will be used for excision. For the classical small-chains argument using subdivision, see [Hat02, Proposition 2.21 and its proof, pp. 119–123].

Definition 7.1.10. A map f between convex subsets of Euclidean spaces is *affine* if it preserves convex combinations: for any points $x_0, \dots, x_m$ in its domain and real numbers $t_i \geq 0$ with $\sum_i t_i = 1$, it satisfies $f(\sum_i t_i x_i) = \sum_i t_i f(x_i)$.

Construction 7.1.11 (Barycentric subdivision). Use the standard topological simplex

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0,\ \textstyle\sum_i t_i = 1\}.$$

The affine face inclusion $d^i : |\Delta^{n-1}| \to |\Delta^n|$ inserts a zero in the ith coordinate.[1] The affine map $s^i : |\Delta^{n+1}| \to |\Delta^n|$ adds the ith and $(i+1)$st coordinates. These maps make $|\Delta^\bullet|$ cosimplicial.

The simplicial set $\mathrm{Mor}_{\mathrm{Top}}(|\Delta^\bullet|, Z)$ has set of n-simplices $\mathrm{Mor}_{\mathrm{Top}}(|\Delta^n|, Z)$. The singular-chain differential is $\partial\sigma = \sum_i (-1)^i \sigma \circ d^i$.

For a convex subset E of a Euclidean space and an affine simplex $\tau : |\Delta^n| \to E$, denote its vertex images by $e_0, \dots, e_n$. The ordered affine simplex $[e_0, \dots, e_n]$ means the map $(t_0, \dots, t_n) \mapsto$

[1] This is the map induced by the simplicial coface under geometric realization; we use the same notation d^i.

$\sum_i t_i e_i$; the order is part of the map. For a nonempty set $I \subseteq \{0, \dots, n\}$, define its *barycenter* by

$$e_I = \frac{1}{|I|} \sum_{i \in I} e_i.$$

In particular, the full barycenter is $e_{0\cdots n} = \tau(1/(n+1), \dots, 1/(n+1))$. This definition uses the indexed vertices and remains valid when some vertex images coincide. Thus e_{02} is the midpoint of the face with indices $0, 2$, and $e_{123} = (e_1 + e_2 + e_3)/3$ is the barycenter of the face with indices $1, 2, 3$. Points e_I and ordered simplices $[e_{i_0}, \dots, e_{i_k}]$ are different kinds of objects.

For a convex subset E of a Euclidean space, work with the augmented chain complex whose degree-n group is the free abelian group

$$\mathbb{Z}\{\tau \in \mathrm{Mor}(|\Delta^n|, E) \mid \tau \text{ is affine}\} \qquad (n \geq 0),$$

with differential induced by the cofaces. Its group in degree -1 is $\mathbb{Z}$, and the augmentation sends each 0-simplex to 1. For a point $e \in E$, coning is the degree-one homomorphism

$$e \star - : \mathbb{Z}\{\tau : |\Delta^n| \to E \text{ affine}\} \longrightarrow \mathbb{Z}\{\tau : |\Delta^{n+1}| \to E \text{ affine}\} \quad (n \geq 0).$$

It joins an affine simplex τ to e by straight line segments, by the formula

$$(e \star \tau)(t_0, \dots, t_{n+1}) = t_0 e + (1 - t_0)\tau\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right),$$

with value e at $t_0 = 1$. In degree -1, the homomorphism $e \star - : \mathbb{Z} \to \mathbb{Z}\{\tau : |\Delta^0| \to E \text{ affine}\}$ sends 1 to the constant 0-simplex at e. Extend the cone formula linearly in every degree.

For $n > 0$, the face of $e \star \tau$ indexed by 0 is τ, and the face indexed by $i+1$ is $e \star (\tau \circ d^i)$. Hence

$$\partial(e \star \tau) = \tau + \sum_{i=0}^{n} (-1)^{i+1} e \star (\tau \circ d^i) = \tau - e \star \partial\tau.$$

For $n = 0$, the boundary is $\tau - (e \star 1)$, and in degree -1 it is $\partial(e \star 1) = 1$. Thus the same formula holds on all augmented chains by linearity.

For every $n \geq 0$, we construct homomorphisms

$$\begin{aligned} \mathrm{Sd} &: \mathbb{Z}\{\tau : |\Delta^n| \to E \text{ affine}\} \longrightarrow \mathbb{Z}\{\tau : |\Delta^n| \to E \text{ affine}\}, \\ T &: \mathbb{Z}\{\tau : |\Delta^n| \to E \text{ affine}\} \longrightarrow \mathbb{Z}\{\tau : |\Delta^{n+1}| \to E \text{ affine}\}, \end{aligned}$$

of degrees zero and one, respectively, with the augmented degree -1 treated below. We define Sd and T inductively. For the base cases, define $\mathrm{Sd} = \mathrm{id}$ and $T = 0$ in degrees -1 and 0. These choices satisfy the required identities: for every chain α of degree 0 or -1,

$$\partial \mathrm{Sd}(\alpha) = \partial\alpha = \mathrm{Sd}(\partial\alpha), \qquad \partial T(\alpha) + T(\partial\alpha) = 0 = \alpha - \mathrm{Sd}(\alpha).$$

Here ∂ in degree zero is the augmentation and in degree -1 is zero.

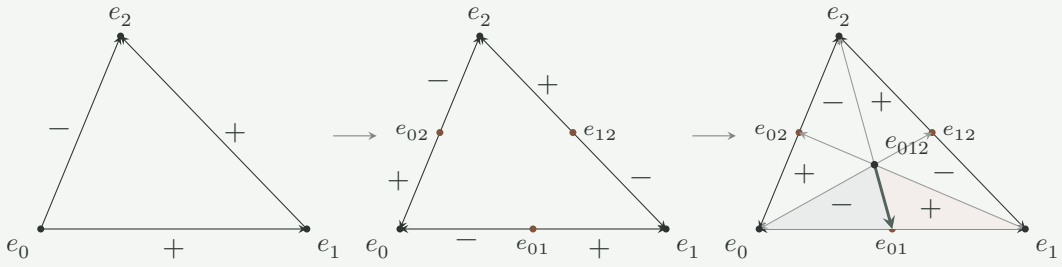
For the inductive step, fix $n > 0$ and assume that Sd and T have been defined in degrees $-1, 0, \dots, n-1$. For every chain α of degree less than n, including degree -1, assume

$$\partial \mathrm{Sd}(\alpha) = \mathrm{Sd}(\partial\alpha), \qquad \partial T(\alpha) + T(\partial\alpha) = \alpha - \mathrm{Sd}(\alpha).$$

For an affine n-simplex τ, put $e = e_{0\cdots n}$, its full barycenter. Define $\mathrm{Sd}(\tau)$ first and then $T(\tau)$ by

$$\mathrm{Sd}(\tau) = e \star \mathrm{Sd}(\partial\tau), \qquad T(\tau) = e \star (\tau - \mathrm{Sd}(\tau) - T(\partial\tau)).$$

For $\tau = [e_0, e_1, e_2]$, the recursion is pictured below, with full barycenter e_{012}. An arrow records the order of the vertices of an affine 1-simplex; its sign is its coefficient in the chain.



For example, the bottom edge contributes $[e_{01}, e_1] - [e_{01}, e_0]$ to $\mathrm{Sd}(\partial\tau)$. Coning gives the two shaded terms $[e_{012}, e_{01}, e_1] - [e_{012}, e_{01}, e_0]$. Their boundaries contain $+[e_{012}, e_{01}]$ and $-[e_{012}, e_{01}]$, so the highlighted interior edge cancels as the *same parametrized simplex* with opposite coefficients. The remaining interior terms cancel in the same way. The six terms are indexed by flags $\{e_i\} \subset [e_i, e_j] \subset \tau$; their vertex order is $[e_{012}, e_{ij}, e_i]$, with the coefficient inherited from $\mathrm{Sd}(\partial\tau)$. Thus the picture expresses the cone identity $\partial(e_{012} \star \mathrm{Sd}(\partial\tau)) = \mathrm{Sd}(\partial\tau) - e_{012} \star \partial\mathrm{Sd}(\partial\tau)$; the next proof verifies that the last term vanishes in every degree.

Proposition 7.1.12. On the augmented chain complex of affine simplices in E from Construction 7.1.11, Sd is a chain map from this complex to itself, and T is a chain homotopy between Sd and the identity. Explicitly,

$$\partial\mathrm{Sd} = \mathrm{Sd}\partial, \qquad \partial T + T\partial = \mathrm{id} - \mathrm{Sd}.$$

Both Sd and T commute with affine maps.

Proof. The identities hold in degrees $-1, 0$ by the computation in the construction. Assume they hold in degrees less than n. Each face $\tau \circ d^i$ has degree $n-1$. The induction hypothesis applies to these faces, without assuming the identity on τ:

$$\begin{aligned}\partial(\mathrm{Sd}(\partial\tau)) &= \sum_{i=0}^{n} (-1)^i \partial(\mathrm{Sd}(\tau \circ d^i)) \\ &= \sum_{i=0}^{n} (-1)^i \mathrm{Sd}(\partial(\tau \circ d^i)) = \mathrm{Sd}(\partial^2\tau) = 0.\end{aligned}$$

The boundary formula for coning gives

$$\partial(e \star \alpha) = \alpha - e \star \partial\alpha, \qquad \partial\alpha = 0 \implies \partial(e \star \alpha) = \alpha.$$

Apply this to $\mathrm{Sd}(\partial\tau)$ and then to $\tau - \mathrm{Sd}(\tau) - T(\partial\tau)$, using the lower-degree homotopy identity:

$$\begin{aligned}\partial\mathrm{Sd}(\tau) &= \mathrm{Sd}(\partial\tau) - e \star \partial\mathrm{Sd}(\partial\tau) = \mathrm{Sd}(\partial\tau), \\ \partial T(\partial\tau) &= (\mathrm{id} - \mathrm{Sd})(\partial\tau) - T(\partial^2\tau) = \partial\tau - \mathrm{Sd}(\partial\tau), \\ \partial(\tau - \mathrm{Sd}(\tau) - T(\partial\tau)) &= \partial\tau - \mathrm{Sd}(\partial\tau) - (\partial\tau - \mathrm{Sd}(\partial\tau)) = 0, \\ \partial T(\tau) &= \partial(e \star (\tau - \mathrm{Sd}(\tau) - T(\partial\tau))) \\ &= (\tau - \mathrm{Sd}(\tau) - T(\partial\tau)) - e \star \partial(\tau - \mathrm{Sd}(\tau) - T(\partial\tau)) \\ &= \tau - \mathrm{Sd}(\tau) - T(\partial\tau).\end{aligned}$$

In the last equality, the term $e \star \partial(\tau - \mathrm{Sd}(\tau) - T(\partial\tau))$ vanishes because the third line shows that the chain in parentheses has zero boundary, and $e \star 0 = 0$ by linearity. That cycle calculation

uses the chain-map identity just proved on τ, the inductive homotopy identity on the lower-degree chain $\partial\tau$, and $\partial^2\tau = 0$. Thus $\partial\mathrm{Sd} = \mathrm{Sd}\partial$ and $\partial T + T\partial = \mathrm{id} - \mathrm{Sd}$ in degree n as well. If $h : E \to E'$ is affine, then $h(e_I) = |I|^{-1}\sum_{i\in I} h(e_i)$ and $h_*(e \star \alpha) = h(e) \star h_*(\alpha)$. These identities, the recursion, and induction on dimension give $h_*\mathrm{Sd}(\tau) = \mathrm{Sd}(h_*\tau)$ and $h_*T(\tau) = T(h_*\tau)$. □

On singular chains, define the degree-zero and degree-one homomorphisms

$$\mathrm{Sd} : \mathrm{C}_n(Z;\mathbb{Z}) \longrightarrow \mathrm{C}_n(Z;\mathbb{Z}), \qquad T : \mathrm{C}_n(Z;\mathbb{Z}) \longrightarrow \mathrm{C}_{n+1}(Z;\mathbb{Z}) \quad (n \geq 0)$$

by the following formulas on each singular n-simplex $\sigma : |\Delta^n| \to Z$: $\mathrm{Sd}(\sigma) = \sigma_*\mathrm{Sd}(\mathrm{id}_{|\Delta^n|})$ and $T(\sigma) = \sigma_*T(\mathrm{id}_{|\Delta^n|})$. Here σ_* means postcomposition of affine simplices in $|\Delta^n|$ with the continuous map σ; σ need not be affine. Put $\iota_n = \mathrm{id}_{|\Delta^n|}$. For $n > 0$, affine naturality for d^i gives

$$\partial\mathrm{Sd}(\sigma) = \sigma_*\mathrm{Sd}(\partial\iota_n) = \sum_{i=0}^{n}(-1)^i(\sigma \circ d^i)_*\mathrm{Sd}(\iota_{n-1}) = \mathrm{Sd}(\partial\sigma),$$
$$\partial T(\sigma) = \sigma_*(\iota_n - \mathrm{Sd}(\iota_n) - T(\partial\iota_n)) = \sigma - \mathrm{Sd}(\sigma) - T(\partial\sigma).$$

For $n = 0$ these equalities hold directly, with the unaugmented singular differential. Every cone used in the recursion stays in the convex hull of its simplex's vertex images. Consequently all terms after postcomposition lie in $\sigma(|\Delta^n|)$, so Sd, T preserve small chains.

For a nonempty subset A of a metric space (K, d_K), its *diameter* is

$$\mathrm{diam}(A) = \sup\{d_K(x,y) : x, y \in A\},$$

with value $+\infty$ when these distances are unbounded.

The vertices of each affine map occurring in $\mathrm{Sd}(\tau)$ are barycenters of nested indexed faces: this holds for a vertex, and the recursion cones each flag in a codimension-one face from the full barycenter, adding the largest face to that flag. For $I \subsetneq J$, put $r = |I|$ and $s = |J|$. Then

$$e_J = \frac{r}{s}e_I + \frac{s-r}{s}e_{J-I}, \qquad \|e_J - e_I\| = \frac{s-r}{s}\|e_{J-I} - e_I\| \leq \frac{n}{n+1}\,\mathrm{diam}(\tau(|\Delta^n|)).$$

Here $1 \leq r < s \leq n+1$, and both barycenters lie in the original convex hull. A convex hull has the same diameter as its vertex set: if $x = \sum_i a_i e_i$ and $y = \sum_j b_j e_j$ are convex combinations, then $\|x - y\| \leq \sum_{i,j} a_i b_j \|e_i - e_j\| \leq \max_{i,j}\|e_i - e_j\|$. Thus each subdivided simplex has diameter at most $n/(n+1)$ times the original diameter. Applying this bound inside each new simplex proves the factor $(n/(n+1))^m$ after m subdivisions for $n > 0$. In dimension zero every simplex is already small.

In the 2-simplex picture, let d be the diameter of τ. Each term has vertices e_{012}, e_{ij}, e_i. The three distances are bounded by

$$\|e_{ij} - e_i\| = \tfrac{1}{2}\|e_j - e_i\| \leq \tfrac{1}{2}d,$$
$$\|e_{012} - e_i\| = \tfrac{2}{3}\|e_{jk} - e_i\| \leq \tfrac{2}{3}d,$$
$$\|e_{012} - e_{ij}\| = \tfrac{1}{3}\|e_k - e_{ij}\| \leq \tfrac{1}{3}d,$$

where $\{i,j,k\} = \{0,1,2\}$. A simplex's diameter is the largest distance between its vertices. Each of the six terms therefore has diameter at most $\frac{2}{3}d$; repeating this estimate inside each term gives the bound $(2/3)^m d$ after m subdivisions. The bound concerns the affine simplices in the domain, before postcomposition with the singular simplex $\sigma : |\Delta^n| \to Z$.

The passage from small diameter to small chains uses the following compactness lemma.

Lemma 7.1.13 (Lebesgue number lemma). Let (K, d_K) be a nonempty compact metric space, and let $\mathcal{U}$ be an open cover of K. There is a number $\delta > 0$ such that every subset of K of diameter less than δ is contained in a member of $\mathcal{U}$.

Proof. For each $x \in K$, choose $\varepsilon_x > 0$ and $U_x \in \mathcal{U}$ with $B(x, 2\varepsilon_x) \subset U_x$. By compactness, finitely many balls $B(x_i, \varepsilon_{x_i})$ cover K. Put $\delta = \min_i \varepsilon_{x_i} > 0$. If a nonempty subset $A \subset K$ has diameter less than δ, choose $a \in A$ and an index i with $a \in B(x_i, \varepsilon_{x_i})$. For every $y \in A$,

$$d_K(y, x_i) \le d_K(y, a) + d_K(a, x_i) < \delta + \varepsilon_{x_i} \le 2\varepsilon_{x_i}.$$

Thus $A \subset B(x_i, 2\varepsilon_{x_i}) \subset U_{x_i}$. The conclusion for $A = \emptyset$ is immediate. □

Apply the Lebesgue number lemma (Lemma 7.1.13) to the cover $\sigma^{-1}(U), \sigma^{-1}(V)$ of the compact metric simplex $|\Delta^n|$. Once the subdivision mesh is less than its Lebesgue number δ, each subdivided domain simplex maps into U or V. Taking sufficiently many subdivisions for the finitely many simplices in a chain α therefore makes $\mathrm{Sd}^m(\alpha)$ small. Define the degree-one homomorphism $T_m : \mathrm{C}_n(Z;\mathbb{Z}) \to \mathrm{C}_{n+1}(Z;\mathbb{Z})$ by $T_m = \sum_{r=0}^{m-1} \mathrm{Sd}^r T$; then

$$\partial T_m + T_m \partial = \mathrm{id} - \mathrm{Sd}^m,$$

and T_m preserves small chains.

We now apply these constructions to small chains.

Theorem 7.1.14 (Small chains). For an open cover $Z = U \cup V$, the inclusion

$$\mathrm{C}_\bullet^{\{U,V\}}(Z) \hookrightarrow \mathrm{C}_\bullet(Z)$$

is a chain homotopy equivalence and hence induces an isomorphism in every homology degree.

Proof. Write $\iota : \mathrm{C}_\bullet^{\{U,V\}}(Z) \hookrightarrow \mathrm{C}_\bullet(Z)$. Construct a degree-zero retraction and a degree-one homomorphism

$$\rho : \mathrm{C}_n(Z) \longrightarrow \mathrm{C}_n^{\{U,V\}}(Z), \qquad D : \mathrm{C}_n(Z) \longrightarrow \mathrm{C}_{n+1}(Z) \quad (n \ge 0)$$

inductively, setting $\rho(\sigma) = \sigma$ and $D(\sigma) = 0$ on small basis simplices. For each remaining basis simplex, put $\alpha = \sigma - D(\partial\sigma)$, choose m making $\mathrm{Sd}^m(\alpha)$ small, and set

$$\iota(\rho(\sigma)) = \mathrm{Sd}^m(\alpha) + T_m(\partial\alpha), \qquad D(\sigma) = T_m(\alpha).$$

All faces of a small simplex are small, so the prescribed values on small simplices satisfy the inductive identities. For a remaining simplex, the lower-dimensional homotopy identity gives

$$\partial\alpha = \partial\sigma - \partial D(\partial\sigma) = \partial\sigma - (\partial\sigma - \iota(\rho(\partial\sigma)) - D(\partial^2\sigma)) = \iota(\rho(\partial\sigma)).$$

Thus both terms defining $\rho(\sigma)$ are small. Moreover,

$$\partial(\mathrm{Sd}^m(\alpha) + T_m(\partial\alpha)) = \mathrm{Sd}^m(\partial\alpha) + (\mathrm{id} - \mathrm{Sd}^m)(\partial\alpha) = \partial\alpha,$$

since $\partial^2\alpha = 0$. The displayed homotopy identity gives

$$\partial\iota(\rho(\sigma)) = \partial\alpha = \iota(\rho(\partial\sigma)), \qquad \partial D(\sigma) = \sigma - D(\partial\sigma) - \iota(\rho(\sigma)).$$

Starting with vertices and extending linearly proves $\rho\iota = \mathrm{id}$, $D\iota = 0$, and $\partial D + D\partial = \mathrm{id} - \iota\rho$. Thus ι is a chain homotopy equivalence. □

Let $f : W \hookrightarrow U$, $g : W \hookrightarrow V$, $i : U \hookrightarrow Z$, and $j : V \hookrightarrow Z$ be the inclusions. There is a short exact sequence of chain complexes

$$0 \longrightarrow \mathrm{C}_\bullet(W) \xrightarrow{(f_*, g_*)} \mathrm{C}_\bullet(U) \oplus \mathrm{C}_\bullet(V) \xrightarrow{j_* - i_*} \mathrm{C}_\bullet^{\{U,V\}}(Z) \longrightarrow 0. \tag{16}$$

The first map is injective because inclusions preserve the basis of singular simplices. In the middle, the equation $\mu - \alpha = 0$ implies that $\alpha = \mu$ is supported on simplices lying in both U and V. This identifies the kernel with the image of the first map. The last map is surjective by definition of small chains.

7.1.4 Algebraic and Geometric Double Mapping Cylinders

The mapping cone gives an explicit homotopy pushout of chain complexes. Its comparison retains the specified homotopy in the original square.

Definition 7.1.15. For chain maps $f_* : \mathrm{C}_\bullet(W) \longrightarrow \mathrm{C}_\bullet(X)$ and $g_* : \mathrm{C}_\bullet(W) \longrightarrow \mathrm{C}_\bullet(Y)$, their *algebraic double mapping cylinder*, or *homotopy pushout*, is the complex $D_\bullet$ with degree n group and differential

$$D_n = \mathrm{C}_n(X) \oplus \mathrm{C}_n(Y) \oplus \mathrm{C}_{n-1}(W),$$
$$\partial_D(\lambda, \mu, \nu) = (\partial\lambda + f_*(\nu),\ \partial\mu + g_*(\nu),\ -\partial\nu).$$

The chain-map identities imply $\partial_{\mathrm{D}}^2 = 0$. The structure maps are $i(\lambda) = (-\lambda, 0, 0)$ and $j(\mu) = (0, \mu, 0)$. The degree-one map $T(\nu) = (0, 0, \nu)$ satisfies

$$\partial_D T + T\partial = jg_* - if_*.$$

For a set S, write $\mathbb{Z}\{S\}$ for the free abelian group on S. For the representable simplicial set Δ^1, use the complex $\mathbb{Z}\{\Delta^1\}^{\mathrm{nd}}_\bullet$: its degree-$n$ group is free abelian on the set of nondegenerate n-simplices of Δ^1. It has generators e_0, e_1 in degree zero and e_{01} in degree one, and differential $\partial e_{01} = e_1 - e_0$. The cylinder chain map

$$h : \mathrm{C}_\bullet(W) \otimes \mathbb{Z}\{\Delta^1\}^{\mathrm{nd}}_\bullet \longrightarrow D_\bullet$$

is given, for $\nu \in \mathrm{C}_n(W)$, by

$$h(\nu \otimes e_0) = if_*(\nu), \qquad h(\nu \otimes e_1) = jg_*(\nu), \qquad h(\nu \otimes e_{01}) = (-1)^n T(\nu).$$

Thus h has endpoints if_* and jg_*, while T is its degree-one component with the displayed sign. With the mapping-cone convention of Lecture 2, this explicitly defined complex is $\mathrm{Cone}(u)$ for $u = (f_*, g_*)$.

Let $f : W \longrightarrow X$ and $g : W \longrightarrow Y$ be continuous maps. Consider the subspaces

$$U = X \underset{W\times\{0\}}{+} (W \times [0, 2/3)), \qquad V = (W \times (1/3, 1]) \underset{W\times\{1\}}{+} Y$$

of $\mathrm{Cyl}(f, g)$, using the identifications in Definition 7.1.1. Their preimages in the defining disjoint union are open and saturated, so U, V are open in $\mathrm{Cyl}(f, g)$. They cover $\mathrm{Cyl}(f, g)$, and $U \cap V$ is $W \times (1/3, 2/3)$.

Let $p_U : U \longrightarrow X$ and $p_V : V \longrightarrow Y$ be the identity on X and Y, respectively, with $p_U([w, t]) = f(w)$ and $p_V([w, t]) = g(w)$ on the cylinder portions. They are deformation retractions, by moving heights linearly to 0 and 1. Projection $p : U \cap V \longrightarrow W$ is a homotopy equivalence. These maps give the commuting diagram of spans

$$\begin{array}{ccccc} U & \hookleftarrow & U \cap V & \hookrightarrow & V \\ {\scriptstyle p_U}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_V} \\ X & \xleftarrow{f} & W & \xrightarrow{g} & Y \end{array}$$

where the vertical maps on the left and right are $p_U : U \longrightarrow X$ and $p_V : V \longrightarrow Y$, respectively.

7.2 Mayer–Vietoris and Homology Calculations

7.2.1 Singular chains preserve homotopy pushouts

Theorem 7.2.1. For every abelian group A, the singular chains functor

$$\mathrm{C}_\bullet(-;A) : \mathrm{Top} \longrightarrow \mathrm{Ch}(\mathrm{Ab})$$

preserves homotopy pushouts.

Remark 7.2.2. Explicitly, for a homotopy pushout square of spaces as in Definition 7.1.3, its comparison from $D_\bullet \otimes_{\mathbb{Z}} A$ to $\mathrm{C}_\bullet(Z;A)$ is a quasi-isomorphism, where $D_\bullet$ is the complex of Definition 7.1.15. In particular, $\mathrm{C}_\bullet(Z)$ is isomorphic to $D_\bullet$ in $\mathcal{D}(\mathbb{Z})$. The chain square induced by any open cover $Z = U \cup V$, with $W = U \cap V$, is also a homotopy pushout.

Proof. First work over $\mathbb{Z}$. For an open cover $Z = U \cup V$, put $W = U \cap V$ and let $\iota : \mathrm{C}_\bullet(W) \to \mathrm{C}_\bullet(U) \oplus \mathrm{C}_\bullet(V)$ be the pair of inclusions. The comparison for the strictly commuting cover square with zero chain homotopy factors as

$$\begin{array}{ccc} \mathrm{Cone}(\iota) & \xrightarrow{\ q\ } & \mathrm{C}_\bullet^{\{U,V\}}(Z) \\ & \searrow & \downarrow \\ & & \mathrm{C}_\bullet(Z). \end{array}$$

In degree n, the cone is

$$\mathrm{Cone}(\iota)_n = \mathrm{C}_n(U) \oplus \mathrm{C}_n(V) \oplus \mathrm{C}_{n-1}(W).$$

Here the three coordinates λ, μ, and ν lie in $\mathrm{C}_n(U)$, $\mathrm{C}_n(V)$, and $\mathrm{C}_{n-1}(W)$, respectively. Writing $i : U \hookrightarrow Z$ and $j : V \hookrightarrow Z$ for the inclusions, the first map is the row matrix

$$\begin{pmatrix} -i_* & j_* & 0 \end{pmatrix} : \mathrm{Cone}(\iota)_n \longrightarrow \mathrm{C}_n^{\{U,V\}}(Z).$$

Write $S_\bullet = \mathrm{C}_\bullet^{\{U,V\}}(Z)$ and denote this map by q. It is surjective: every generating small simplex lies in U or V, and is the image of $(-\sigma, 0, 0)$ or $(0, \sigma, 0)$, respectively. It is a chain map because the two inclusions of W into Z agree, so the contributions of ν cancel in $q\partial$.

Identify chains on U, V, and W with their images in chains on Z. The common simplex basis gives

$$\mathrm{C}_n(U) \cap \mathrm{C}_n(V) = \mathrm{C}_n(W).$$

Thus, writing $K_\bullet = \ker(q)$,

$$\begin{aligned} K_n &= \{(a,a,b) : a \in \mathrm{C}_n(W),\ b \in \mathrm{C}_{n-1}(W)\}, \\ d_K(a,a,b) &= (\partial a + b, \partial a + b, -\partial b), \\ s_n : K_n &\longrightarrow K_{n+1}, \qquad (a,a,b) \longmapsto (0,0,a). \end{aligned}$$

The contraction is verified in the kernel itself:

$$\begin{aligned} d_K s_n(a,a,b) &= (a,a,-\partial a), \\ s_{n-1} d_K(a,a,b) &= (0,0,\partial a + b), \\ (d_K s_n + s_{n-1} d_K)(a,a,b) &= (a,a,b). \end{aligned}$$

Hence s is a chain homotopy from 0 to $\mathrm{id}_{K_\bullet}$: $K_\bullet$ is contractible. In particular, every cycle (a,a,b) satisfies $b = -\partial a$ and is the boundary $d_K(0,0,a)$. The long exact homology sequence of

$$0 \longrightarrow K_\bullet \longrightarrow \mathrm{Cone}(\iota) \xrightarrow{q} S_\bullet \longrightarrow 0$$

now proves that q is a quasi-isomorphism. The second map is a chain homotopy equivalence by Theorem 7.1.14. This proves the open-cover assertion over $\mathbb{Z}$.

Apply the open-cover assertion to the cover of $\mathrm{Cyl}(f,g)$ constructed above. The retractions p_U, p_V and projection p give a strictly commuting diagram

$$\begin{array}{ccccc} \mathrm{C}_\bullet(U) & \longleftarrow & \mathrm{C}_\bullet(U\cap V) & \longrightarrow & \mathrm{C}_\bullet(V) \\ \downarrow{\scriptstyle p_{U,*}} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_{V,*}} \\ \mathrm{C}_\bullet(X) & \xleftarrow{f_*} & \mathrm{C}_\bullet(W) & \xrightarrow{g_*} & \mathrm{C}_\bullet(Y). \end{array}$$

The vertical maps are chain homotopy equivalences by homotopy invariance of singular chains, since the corresponding maps of spaces were proved to be homotopy equivalences above. Homotopy invariance of derived pushouts therefore identifies the derived pushouts of these two spans. The upper one is $\mathrm{C}_\bullet(\mathrm{Cyl}(f,g))$ by the open-cover assertion; the lower one is $D_\bullet$ by Definition 7.1.15.

This identification respects the specified cylinder homotopy $\eta(w,t) = [w,t]$. Indeed, on the overlap the contraction in U, read backwards, gives the path from $[w,0]$ to $[w,t]$, and the contraction in V gives the path from $[w,t]$ to $[w,1]$. Their concatenation is the cylinder path with a break at height t. Linear interpolation of its height parametrization with the identity gives a homotopy relative to the endpoints, continuously in (w,t). Thus the transported cover cocone is homotopic to the cocone determined by η, so its derived comparison from $D_\bullet$ to $\mathrm{C}_\bullet(\mathrm{Cyl}(f,g))$ is an isomorphism.

For the given homotopy pushout square, the map $\theta_h : \mathrm{Cyl}(f,g) \to Z$ is a homotopy equivalence and satisfies $\theta_h \eta = h$. Homotopy invariance of singular chains and composition with $\theta_{h,*}$ now identify the comparison for the specified homotopy h with an isomorphism in $\mathcal{D}(\mathbb{Z})$. Hence that comparison is a quasi-isomorphism, proving preservation over $\mathbb{Z}$.

Every integral complex used here is nonnegative and degreewise free. A quasi-isomorphism between such complexes is a chain homotopy equivalence: its acyclic cone is contractible, since its cycle groups are free by Lecture 3, Theorem 3.1.10, and its sequences of cycles and boundaries split. In the splitting $C_n = Z_n \oplus Z_{n-1}$ of an acyclic free complex, the differential is $(z,w) \mapsto (w,0)$ and the contracting homotopy is $(z,w) \mapsto (0,z)$. Tensoring with A preserves chain homotopies and commutes with mapping cones. It therefore preserves these homotopy pushouts and gives the result for $\mathrm{C}_\bullet(-;\mathbb{Z}) \otimes_{\mathbb{Z}} A$, without a flatness hypothesis on A. Equivalently, the free complexes compute derived tensoring. □

7.2.2 Mayer–Vietoris and Excision

The preservation theorem gives long exact sequences and relative comparisons by applying the corresponding constructions in chain complexes.

Corollary 7.2.3. For every abelian group A, a homotopy pushout square

$$\begin{array}{ccc} & X & \\ {\scriptstyle f}\nearrow & \Downarrow h & \searrow{\scriptstyle i} \\ W & & Z \\ {\scriptstyle g}\searrow & & \nearrow{\scriptstyle j} \\ & Y & \end{array}$$

as in Definition 7.1.3 induces a natural long exact sequence. Five consecutive terms are

$$\mathrm{H}_n(W;A) \xrightarrow{(f_*,g_*)} \mathrm{H}_n(X;A)\oplus \mathrm{H}_n(Y;A) \xrightarrow{j_*-i_*} \mathrm{H}_n(Z;A) \xrightarrow{\delta_n^Z} \mathrm{H}_{n-1}(W;A) \xrightarrow{(f_*,g_*)} \mathrm{H}_{n-1}(X;A)\oplus \mathrm{H}_{n-1}(Y;A). \quad (17)$$

The sequence ends with $\mathrm{H}_0(Z;A) \longrightarrow 0$. It is natural for maps preserving the square and its specified homotopy.

Proof. Apply the mapping-cone long exact sequence of Lecture 2 to the homotopy pushout of Theorem 7.2.1. Its cone triangle is

$$\mathrm{C}_\bullet(W;A) \xrightarrow{(f_*,g_*)} \mathrm{C}_\bullet(X;A) \oplus \mathrm{C}_\bullet(Y;A) \longrightarrow D_\bullet \otimes_{\mathbb{Z}} A \longrightarrow \mathrm{C}_\bullet(W;A)[1].$$

The comparison identifies the middle map with $j_* - i_*$. Taking the negative of the shifted projection as connecting map gives the stated convention. The shift has differential $-\partial$, and all constructions are natural in the specified square. □

Corollary 7.2.4 (Mayer–Vietoris). For an open cover $Z = U \cup V$, put $W = U \cap V$. Let f, g be the inclusions of W into U, V, and let i, j be the inclusions of U, V into Z, respectively. For every abelian group A, they induce a natural long exact sequence. Five consecutive terms are

$$\mathrm{H}_n(W;A) \xrightarrow{(f_*,g_*)} \mathrm{H}_n(U;A) \oplus \mathrm{H}_n(V;A) \xrightarrow{j_*-i_*} \mathrm{H}_n(Z;A) \xrightarrow{\delta_n} \mathrm{H}_{n-1}(W;A) \xrightarrow{(f_*,g_*)} \mathrm{H}_{n-1}(U;A) \oplus \mathrm{H}_{n-1}(V;A).$$

The sequence ends with $\mathrm{H}_0(Z;A) \longrightarrow 0$. If a class is represented by a small cycle $\zeta = \mu - \lambda$, with $\lambda \in \mathrm{C}_n(U;A)$ and $\mu \in \mathrm{C}_n(V;A)$, then $\delta_n[\zeta] = [\partial\lambda] = [\partial\mu]$ in $\mathrm{H}_{n-1}(W;A)$.

Proof. Apply the mapping-cone long exact sequence to the open-cover assertion of Theorem 7.2.1. To compute its connecting map, use the degreewise split sequence (16) tensored with A. A lift (λ, μ) of $\zeta = \mu - \lambda$ has boundary $(\partial\lambda, \partial\mu) = (\partial\lambda, \partial\lambda)$, so $\delta_n[\zeta] = [\partial\lambda]$. This agrees with the negative shifted projection in Corollary 7.2.3. Naturality and independence of the lift follow from the long exact sequence construction. □

The second map means $(\lambda, \mu) \mapsto j_*(\mu) - i_*(\lambda)$. Its composite with (f_*, g_*) is zero because $if = jg$.

Corollary 7.2.5 (Excision). For an open cover $Z = U \cup V$, put $W = U \cap V$. For every abelian group A, the inclusions induce isomorphisms

$$\mathrm{H}_n(V, W;A) \xrightarrow{\cong} \mathrm{H}_n(Z, U;A).$$

Proof. Write $S_\bullet = \mathrm{C}_\bullet^{\{U,V\}}(Z;\mathbb{Z})$. The maps ρ, D in Theorem 7.1.14 satisfy $\rho\iota = \mathrm{id}$, $D\iota = 0$, and $\partial D + D\partial = \mathrm{id} - \iota\rho$. In particular, ρ fixes $\mathrm{C}_\bullet(U;\mathbb{Z})$ and D vanishes there. Hence these maps descend to quotients and exhibit

$$S_\bullet/\mathrm{C}_\bullet(U;\mathbb{Z}) \longrightarrow \mathrm{C}_\bullet(Z;\mathbb{Z})/\mathrm{C}_\bullet(U;\mathbb{Z})$$

as a chain homotopy equivalence. The map from $\mathrm{C}_\bullet(V;\mathbb{Z})$ onto the left quotient is surjective because every small simplex lies in U or V. Its kernel consists exactly of chains supported on simplices lying in both U and V, namely $\mathrm{C}_\bullet(W;\mathbb{Z})$. It therefore identifies that quotient with $\mathrm{C}_\bullet(V, W;\mathbb{Z})$, and the resulting comparison is the inclusion-induced relative chain map:

$$\begin{array}{ccc} \mathrm{C}_\bullet(V,W;\mathbb{Z}) & \xrightarrow{\cong} & S_\bullet/\mathrm{C}_\bullet(U;\mathbb{Z}) \\ & \searrow^{\text{inclusion}} & \downarrow \simeq \\ & & \mathrm{C}_\bullet(Z,U;\mathbb{Z}). \end{array}$$

These relative-chain quotients are degreewise free on the remaining singular simplices. Tensoring the maps and homotopy identities with A identifies their quotients with the corresponding relative chains with coefficients in A and retains the chain homotopy equivalence. Taking homology proves the assertion. □

7.2.3 The Gluing Axiom and Its Meaning

The Mayer–Vietoris axiom of Lecture 5 is the abstract form of Corollary 7.2.3. Suspensions, cell attachments, and mapping cones give its basic computations.

Let $A_n : \mathrm{Top} \longrightarrow \mathrm{Ab}$, for $n \in \mathbb{Z}$, be functors equipped with natural Mayer–Vietoris connecting maps for homotopy pushout squares. Assume homotopy invariance, additivity, and exactness of the Mayer–Vietoris sequences. Homotopy invariance sends homotopy equivalences to isomorphisms; additivity identifies $A_n(X+Y)$ with $A_n(X) \oplus A_n(Y)$.

Proposition 7.2.6. For a homotopy pushout square of topological spaces

$$\begin{array}{ccc} & X & \\ {\scriptstyle f}\nearrow & \Downarrow h & \searrow{\scriptstyle i} \\ W & & Z \\ {\scriptstyle g}\searrow & & \nearrow{\scriptstyle j} \\ & Y & \end{array}$$

for every integer n, five consecutive terms of the long exact sequence are

$$A_n(W) \xrightarrow{(f_*,g_*)} A_n(X) \oplus A_n(Y) \xrightarrow{j_*-i_*} A_n(Z) \xrightarrow{\delta_n^Z} A_{n-1}(W) \xrightarrow{(f_*,g_*)} A_{n-1}(X) \oplus A_{n-1}(Y).$$

If $\delta_n^{\mathrm{Cyl}(f,g)}$ is the connecting map for the span (f,g), then $\delta_n^Z = \delta_n^{\mathrm{Cyl}(f,g)}(\theta_{h,*})^{-1}$.

Proof. The Mayer–Vietoris sequence of Lecture 5 is put in the displayed convention by multiplying the $A_n(Y)$ summand and the $A_n(\mathrm{Cyl}(f,g))$ term by -1. Thus the cylinder connecting map here is the negative of the one in Lecture 5. Homotopy invariance makes $\theta_{h,*}$ invertible, so transport that sequence to $A_n(Z)$. The identities on the two restrictions of θ_h identify the adjacent homomorphism with $j_* - i_*$. Naturality follows for compatible maps of the chosen squares, as in Lecture 5. □

Remark 7.2.7. Degreewise homology groups need not form a pushout square in Ab. The ordinary pushout of $A_n(X) \leftarrow A_n(W) \rightarrow A_n(Y)$ is canonically isomorphic to the cokernel of (f_*, g_*), by changing the sign on the $A_n(X)$ summand. Exactness in Proposition 7.2.6 identifies this cokernel with $\ker(\delta_n^Z)$, a subgroup of $A_n(Z)$. It is all of $A_n(Z)$ if and only if $\delta_n^Z = 0$. For the suspension square of S^0 in Example 7.1.4, ordinary integral homology in degree one vanishes at the other three corners but $\mathrm{H}_1(\mathrm{S}^1;\mathbb{Z}) = \mathbb{Z}$. The preservation statement is the long exact sequence, or the derived pushout of chains in Theorem 7.2.1.

Use reduced generalized homology and its basepoint splitting from Lecture 5, Definition 5.3.1. Pointed maps preserve this splitting. For ordinary homology the coefficient summand is $\mathrm{H}_n(*;\mathbb{Z})$, computed in Lecture 6, Proposition 6.2.2.

Proposition 7.2.8. Let W be a nonempty pointed topological space. For its unreduced suspension $\mathrm{S}(W)$, pointed at the top vertex, the connecting map gives isomorphisms

$$\widetilde{A}_n(\mathrm{S}(W)) \xrightarrow{\cong} \widetilde{A}_{n-1}(W) \qquad (n \in \mathbb{Z}).$$

Proof. Apply Proposition 7.2.6 to $* \xleftarrow{\mathrm{pr}_W} W \xrightarrow{\mathrm{pr}_W} *$. In degree $n-1$, the map from $A_{n-1}(W)$ to the two point groups is $((\mathrm{pr}_W)_*, (\mathrm{pr}_W)_*)$, whose kernel is $\widetilde{A}_{n-1}(W)$. Exactness makes this kernel the image of δ_n.

The map from the two point groups into $A_n(\mathrm{S}(W))$ is the second vertex map minus the first. These vertices are joined by the interval over the basepoint of W, so their maps on homology agree. The image of their difference on the direct sum is the coefficient summand

$A_n(*)$ in $A_n(\mathrm{S}(W))$. Exactness therefore identifies $A_n(\mathrm{S}(W))/A_n(*)$ with $\widetilde{A}_{n-1}(W)$. The basepoint splitting identifies this quotient with $\widetilde{A}_n(\mathrm{S}(W))$, proving the assertion. □

The sphere groups and their coefficient dependence are already computed in Lecture 5, Proposition 5.3.6 and Example 5.3.7. Proposition 7.2.8 supplies the suspension interpretation used in the cell calculations below.

7.2.4 Cell Attachments and Mapping Cones

The exact sequence retains both the gluing maps and the connecting map, so it determines which classes survive an attachment and which new classes occur.

Example 7.2.9. Cover S^1 by the complements U, V of two distinct points. Each is an open interval and $W = U \cap V$ has two interval components. The sequence of Corollary 7.2.4 contains

$$0 \longrightarrow \mathrm{H}_1(\mathrm{S}^1;\mathbb{Z}) \xrightarrow{\delta_1} \mathbb{Z}^2 \xrightarrow{(f_*,g_*)} \mathbb{Z}^2 \longrightarrow \mathrm{H}_0(\mathrm{S}^1;\mathbb{Z}) \longrightarrow 0.$$

Each component of W maps to the unique component of each cover member. Thus the middle map sends (u,v) to $(u+v,u+v)$. Its kernel is generated by $(1,-1)$, so $\mathrm{H}_1(\mathrm{S}^1;\mathbb{Z}) \cong \mathbb{Z}$. Its cokernel is $\mathbb{Z}$, in agreement with connectedness. Splitting an oriented circuit into its two arcs gives a boundary equal to the difference of the two overlap points; this is the formula for δ_1 in Corollary 7.2.4.

Proposition 7.2.10. Let $m \geq 2$ and let $f : \mathrm{S}^{m-1} \longrightarrow X$ attach an m-cell, producing $Z = X \underset{\mathrm{S}^{m-1}}{+} \mathrm{D}^m$. There is an exact sequence

$$0 \to \mathrm{H}_m(X;\mathbb{Z}) \to \mathrm{H}_m(Z;\mathbb{Z}) \to \mathbb{Z} \xrightarrow{f_*} \mathrm{H}_{m-1}(X;\mathbb{Z}) \to \mathrm{H}_{m-1}(Z;\mathbb{Z}) \to 0.$$

The homomorphism f_* sends a chosen generator of $\mathrm{H}_{m-1}(\mathrm{S}^{m-1};\mathbb{Z})$ to the homology class of the attaching map. In positive degrees other than $m, m-1$, the inclusion induces $\mathrm{H}_n(X;\mathbb{Z}) \cong \mathrm{H}_n(Z;\mathbb{Z})$; it also induces an isomorphism on $\mathrm{H}_0(-;\mathbb{Z})$.

Proof. Example 7.1.8 gives a homotopy pushout. Apply Corollary 7.2.3, using the sphere homology from Lecture 5, Proposition 5.3.6 and the contractibility of D^m. In the relevant positive degrees, the disk groups vanish and $\mathrm{H}_{m-1}(\mathrm{S}^{m-1};\mathbb{Z}) = \mathbb{Z}$, giving the displayed sequence. The remaining positive-degree terms vanish on the sphere and disk, giving the stated isomorphisms.

Since $m \geq 2$, the attaching sphere is connected. In degree zero the map $\mathrm{H}_0(\mathrm{S}^{m-1};\mathbb{Z}) \longrightarrow \mathrm{H}_0(X;\mathbb{Z}) \oplus \mathrm{H}_0(\mathrm{D}^m;\mathbb{Z})$ identifies the disk component with the component of X containing the image of f. Its cokernel is $\mathrm{H}_0(X;\mathbb{Z})$, proving the final assertion. □

For a pointed map $f : W \longrightarrow X$ between nonempty pointed spaces, Example 7.1.9 and Proposition 7.2.6 also give the reduced cone long exact sequence, with five consecutive terms

$$\widetilde{A}_n(W) \xrightarrow{f_*} \widetilde{A}_n(X) \xrightarrow{-i_*} \widetilde{A}_n(\mathrm{Cone}(f)) \longrightarrow \widetilde{A}_{n-1}(W) \xrightarrow{f_*} \widetilde{A}_{n-1}(X).$$

Here $i : X \longrightarrow \mathrm{Cone}(f)$ is the canonical inclusion. To see the reduction, split off the point groups in the unreduced Mayer–Vietoris sequence. On those summands the maps are the split exact sequence $A_n(*) \xrightarrow{u \mapsto (u,u)} A_n(*)^2 \xrightarrow{(u,v) \mapsto v-u} A_n(*)$. The cone on the basepoint gives the homotopy identifying the two point inclusions, and its connecting map is zero. The remaining summands are the displayed reduced sequence. If f is a Hurewicz cofibration, it computes $\widetilde{A}_n(X/W)$ by Proposition 7.1.6.

References

[Hat02] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002. See also §2.2, pp. 149–150 for mapping tori.

[May99] J. P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press, 1999. Further background for Lectures 5 and 7: Chapters 6, 10, and 13–15; Chapter 10, §7, p. 80 treats the double mapping cylinder.

6.1.2 Cutting the Prism Into Simplices

Fix the base dimension $n \geq 0$. For each prism index $j = 0, \dots, n$, the jth prism simplex has the ordered vertices

$$(e_0, 0), \dots, (e_j, 0), (e_j, 1), \dots, (e_n, 1).$$

Write $s_i = (e_i, 0)$ for the bottom vertices. The affine parametrization is

$$\tau_j : |\Delta^{n+1}| \longrightarrow |\Delta^n| \times I, \qquad \tau_j(e_k) = \begin{cases} (e_k, 0), & 0 \leq k \leq j, \\ (e_{k-1}, 1), & j+1 \leq k \leq n+1. \end{cases}$$

Here k is a vertex position in the prism simplex. The e_k on the left is a standard vertex of the source; those on the right are standard vertices of the base. Affine extension gives the coordinate formula

$$\tau_j : |\Delta^{n+1}| \longrightarrow |\Delta^n| \times I,$$
$$\tau_j(t_0, \dots, t_{n+1}) = \left(\sum_{k=0}^{j} t_k e_k + \sum_{k=j+1}^{n+1} t_k e_{k-1}, \ \sum_{k=j+1}^{n+1} t_k \right).$$

Thus τ_j identifies the two source coordinates in positions $j, j+1$ in the base, and adds the coordinates after position j to obtain the height.

Remark 6.1.4. For a point (x, t) in the prism, now write $x = \sum_{i=0}^{n} t_i e_i$. It belongs to the jth piece exactly when

$$\sum_{i=j+1}^{n} t_i \ \leq t \ \leq \sum_{i=j}^{n} t_i. \tag{15}$$

Indeed, the unique coordinates in the source of τ_j are

$$\left(t_0, \dots, t_{j-1}, \ \sum_{i=j}^{n} t_i - t, \ t - \sum_{i=j+1}^{n} t_i, \ t_{j+1}, \dots, t_n \right).$$

They sum to 1, and they are nonnegative exactly under (15). An empty sum is zero, and an empty string of coordinates is omitted.

For a fixed base point, the endpoints of the intervals in (15) occur in order:

$$0 \leq t_n \leq t_{n-1} + t_n \leq \cdots \leq t_0 + \cdots + t_n = 1.$$

Hence the pieces cover the prism. For $j < j'$, membership in both pieces forces

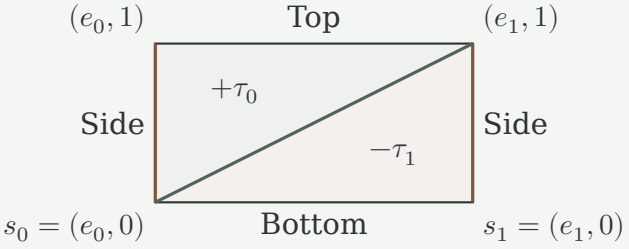
$$t = \sum_{i=j+1}^{n} t_i = \sum_{i=j'}^{n} t_i, \qquad t_{j+1} = \cdots = t_{j'-1} = 0.$$

Their intersection is exactly the common face whose ordered vertices are

$$(e_0, 0), \dots, (e_j, 0), (e_{j'}, 1), \dots, (e_n, 1).$$

Here a face means all linear combinations of the listed vertices with nonnegative coefficients summing to 1. Thus the pieces meet along faces and form a triangulation.

For $n = 1$ the picture is a square divided into two triangles:



The open diagonal is inside the square. The vertical edges lie over the boundary points of the base interval. This distinction persists in every dimension.

Definition 6.1.5. For a singular simplex σ, define σ_j by the composite

$$|\Delta^{n+1}| \xrightarrow{\tau_j} |\Delta^n| \times I \xrightarrow{\sigma \times \mathrm{id}_I} X \times I \xrightarrow{h} Y.$$

Explicitly, the middle map is

$$\sigma \times \mathrm{id}_I : |\Delta^n| \times I \longrightarrow X \times I, \qquad (x,t) \longmapsto (\sigma(x), t).$$

The free abelian universal property extends the following assignment uniquely to a homomorphism:

$$P_n : \mathrm{C}_n(X) \longrightarrow \mathrm{C}_{n+1}(Y), \qquad \sigma \longmapsto \sum_{j=0}^{n} (-1)^j \sigma_j.$$

6.1.3 Prism Faces and Their Cancellation

The boundary of the signed prism consists of its two ends and the prisms on its base faces. This identity gives homotopy invariance and the first absolute computations.

The terms in the boundary are indexed by a Cartesian product:

$$(j,k) \in \{0,\dots,n\} \times \{0,\dots,n+1\}, \qquad \partial_{n+1} P_n(\sigma) = \sum_{j=0}^{n} \sum_{k=0}^{n+1} (-1)^{j+k} \sigma_j d^k.$$

The first index chooses the prism piece; the second deletes its vertex in position k. Each corresponding affine face has type

$$\tau_j d^k : |\Delta^n| \longrightarrow |\Delta^n| \times I.$$

The base vertex e_j appears twice in the piece, in positions j and $j+1$. Every other base vertex appears once.

Remark 6.1.6. For $\epsilon \in \{0,1\}$ the endpoint inclusion of the base is

$$\iota_\epsilon : |\Delta^n| \longrightarrow |\Delta^n| \times I, \qquad x \longmapsto (x, \epsilon).$$

The equalities of affine maps, including their signs, are

$$\tau_0 d^0 = \iota_1 : |\Delta^n| \longrightarrow |\Delta^n| \times I, \qquad (-1)^{0+0} = +1,$$

$$\tau_n d^{n+1} = \iota_0 : |\Delta^n| \longrightarrow |\Delta^n| \times I, \qquad (-1)^{n+n+1} = -1.$$

After composition with the homotopy, these give $g\sigma - f\sigma$.

Remark 6.1.7. For $0 \le j < n$, the two faces

$$\tau_j d^{j+1} = \tau_{j+1} d^{j+1} : |\Delta^n| \longrightarrow |\Delta^n| \times I$$

have the same ordered vertices:

$$(e_0,0), \dots, (e_j,0), (e_{j+1},1), \dots, (e_n,1).$$