Computations from the Eilenberg–Steenrod Axioms

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8.1 Cell Complexes and Degree

We describe cell complexes by successive attachments. Write D^n for the closed unit disk and $\mathrm{S}^{n-1} = \partial \mathrm{D}^n$; in dimension zero, $\mathrm{D}^0 = *$ and $\mathrm{S}^{-1} = \emptyset$.

8.1.1 Cell Attachments and Skeletal Quotients

Definition 8.1.1. To *attach n-cells* to a space X, choose a set Λ and a continuous map $f : W \to X$, where $W = \coprod_{\lambda \in \Lambda} \mathrm{S}^{n-1}$. Let $Y = \coprod_{\lambda \in \Lambda} \mathrm{D}^n$ and let $g : W \to Y$ be the boundary inclusions. The resulting space is the pushout

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{j} & Z. \end{array}$$

Thus $Z = (X + Y)/\!\sim$, with the quotient topology and identifications $f(w) \sim g(w)$ for $w \in W$. The restrictions $f_\lambda : \mathrm{S}^{n-1} \to X$ are the *attaching maps*; the restrictions $j_\lambda : \mathrm{D}^n \to Z$ are the *characteristic maps*. Each j_λ restricts to a homeomorphism from the disk interior onto an *open n-cell*. For $n = 0$, this construction adds one isolated point for each $\lambda \in \Lambda$.

Definition 8.1.2. A *cell complex* is a directed colimit

$$X = \operatorname*{colim}_{n \geq 0} \mathrm{sk}_n X \quad \text{in Top}, \qquad \emptyset = \mathrm{sk}_{-1} X \subseteq \mathrm{sk}_0 X \subseteq \mathrm{sk}_1 X \subseteq \cdots,$$

where $\mathrm{sk}_n X$ is obtained from $\mathrm{sk}_{n-1} X$ by attaching n-cells as in Definition 8.1.1. The colimit supplies the topology on X. Its *n-skeleton* $\mathrm{sk}_n X$ is the union of its cells of dimension at most n, and $\mathrm{sk}_n X = \emptyset$ for $n < 0$. The complex is *finite* if it has finitely many cells in total; then $X = \mathrm{sk}_m X$ for some m. A characteristic map need not be injective on its boundary.

Remark 8.1.3. The homology calculations below concern finite cell complexes. All characteristic maps, including those whose boundary maps are not injective, are interpreted through the attaching pushout.

Proposition 8.1.4. Let Λ_n index the n-cells of a finite cell complex. For $n \geq 0$, the pointed quotient satisfies

$$\frac{(\mathrm{sk}_n X)_+}{(\mathrm{sk}_{n-1} X)_+} \cong \bigvee_{\lambda \in \Lambda_n} \mathrm{S}^n.$$

The empty wedge is $*$. When $\mathrm{sk}_{n-1} X \neq \emptyset$, the left side is the usual quotient $\mathrm{sk}_n X / \mathrm{sk}_{n-1} X$; when $n = 0$, it is $(\mathrm{sk}_0 X)_+$.

Proof. Collapse the previous skeleton and the added point in the attaching quotient. Each n-disk then has its entire boundary collapsed to the common basepoint, and its interior remains unchanged. For $n > 0$ this gives one $\mathrm{D}^n/\mathrm{S}^{n-1} \cong \mathrm{S}^n$ per cell. For $n = 0$ each new point together with the common basepoint is S^0. If there are no cells, only the basepoint remains. Both descriptions impose the same identifications, so the quotient universal property gives the homeomorphism. □

8.1.2 Finite Product Cell Structures

Products of characteristic maps give the cells used in Lecture 8.

Proposition 8.1.5. If X and Y are finite cell complexes, their ordinary product has a finite cell structure whose cells are $e \times e'$, of dimension $p+q$ when e and e' have dimensions p and q. Its skeleta are

$$\mathrm{sk}_n(X \times Y) = \bigcup_{p+q \le n} \mathrm{sk}_p X \times \mathrm{sk}_q Y \qquad (p, q \ge 0).$$

Proof. Take products of characteristic maps. The pair consisting of $\mathrm{D}^p \times \mathrm{D}^q$ and its boundary is homeomorphic to $(\mathrm{D}^{p+q}, \mathrm{S}^{p+q-1})$. Its boundary is

$$(\mathrm{S}^{p-1} \times \mathrm{D}^q) \cup (\mathrm{D}^p \times \mathrm{S}^{q-1}),$$

where the two faces are glued along $\mathrm{S}^{p-1} \times \mathrm{S}^{q-1}$. Both faces map into the preceding skeleton. The resulting finite attachment space maps continuously and bijectively to $X \times Y$. Each finite cell complex is compact Hausdorff: this holds inductively when finitely many compact disks are attached along their closed boundaries. Thus the map is a homeomorphism, which verifies the ordinary product topology. □

8.1.3 Degree of Sphere Maps

By Lecture 5, Proposition 5.3.6, the reduced integral homology of an n-sphere is infinite cyclic in degree n. This gives the following invariant of maps between spheres.

Definition 8.1.6. For $n \ge 1$, a *homological orientation* of S^n is a choice of generator of $\widetilde{\mathrm{H}}_n(\mathrm{S}^n; \mathbb{Z}) \cong \mathbb{Z}$. After choosing such generators for the source and target, the *degree* of a map $f : \mathrm{S}^n \to \mathrm{S}^n$ is the integer $\deg(f)$ by which its induced homomorphism multiplies them.

Throughout the following two propositions, let $n \ge 1$. When source and target are the same oriented sphere, reversing its orientation on both sides does not change this integer. Reversing just one of the two orientations changes its sign.

Proposition 8.1.7. For maps $f, g : \mathrm{S}^n \to \mathrm{S}^n$, one has $\deg(gf) = \deg(g)\deg(f)$. Homotopic maps have the same degree. The identity has degree 1, a constant map has degree 0, and a homotopy equivalence has degree 1 or -1. Every map of nonzero degree is surjective.

Proof. Functoriality of integral homology proves the composition formula and the assertion about the identity. Homotopy invariance proves the second assertion. A constant map factors through a point, whose positive-dimensional integral homology vanishes. If g is a homotopy inverse to f, then $\deg(g)\deg(f) = 1$ in $\mathbb{Z}$, so each integer is 1 or -1.

If a point y is absent from the image, then f factors through $\mathrm{S}^n - \{y\}$. Stereographic projection identifies this complement with the contractible space $\mathbb{R}^n$. The induced map on $\mathrm{H}_n(-; \mathbb{Z})$ is therefore zero. This proves the final assertion by contraposition. □

Proposition 8.1.8. If $f : \mathrm{S}^n \to \mathrm{S}^n$ extends to a continuous map $F : \mathrm{D}^{n+1} \to \mathrm{S}^n$, then $\deg(f) = 0$.

Proof. The extension gives the commutative triangle

$$\begin{array}{ccc} \mathrm{S}^n & \xrightarrow{\ i\ } & \mathrm{D}^{n+1} \\ & \searrow^{f} & \downarrow{\scriptstyle F} \\ & & \mathrm{S}^n. \end{array}$$

After applying $\mathrm{H}_n(-;\mathbb{Z})$, the map f_* factors through the zero group $\mathrm{H}_n(\mathrm{D}^{n+1};\mathbb{Z})$. Hence its degree is zero. □

This is a necessary condition for extension. Its proof uses only homology and does not require a classification of sphere maps up to homotopy. It is the obstruction that will be used in the proof of Brouwer's fixed-point theorem.

Example 8.1.9. Attach an m-cell to S^{m-1} along a map of degree d, with $m \geq 2$. Lecture 7, Proposition 7.2.10 gives

$$\mathrm{H}_m(\mathrm{S}^{m-1} \underset{\mathrm{S}^{m-1}}{+} \mathrm{D}^m;\mathbb{Z}) = \ker(d : \mathbb{Z} \longrightarrow \mathbb{Z}), \qquad \mathrm{H}_{m-1}(\mathrm{S}^{m-1} \underset{\mathrm{S}^{m-1}}{+} \mathrm{D}^m;\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}.$$

For $d \neq 0$ these groups are 0 and $\mathbb{Z}/d\mathbb{Z}$; for $d = 0$ both are $\mathbb{Z}$. All other reduced homology groups vanish. For $m = 2$ and $d = 2$, this is the disk attachment describing $\mathbb{R}P^2$; its homology was computed in Lecture 4, Example 4.4.12.

The following degree computations use homological orientations throughout. They will determine the attaching coefficients of real projective spaces and provide explicit maps of every integral degree.

Lemma 8.1.10. For $n \geq 1$, suspension preserves the degree of maps $\mathrm{S}^n \to \mathrm{S}^n$. Every coordinate reflection of $\mathrm{S}^n \subseteq \mathbb{R}^{n+1}$ has degree -1, and its antipodal map has degree $(-1)^{n+1}$. For every integer m, the map $p_m : \mathrm{S}^1 \to \mathrm{S}^1$ defined by $p_m(z) = z^m$ has degree m. Its iterated suspensions give pointed degree-m maps in every positive dimension.

Proof. Let $f : \mathrm{S}^n \to \mathrm{S}^n$ with $n \geq 0$. For ordinary reduced integral homology, naturality of the connecting map in Lecture 7, Proposition 7.2.8 gives the commutative square

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_{n+1}(\mathrm{S}(\mathrm{S}^n);\mathbb{Z}) & \xrightarrow[\cong]{\delta} & \widetilde{\mathrm{H}}_n(\mathrm{S}^n;\mathbb{Z}) \\ {\scriptstyle (\mathrm{S}(f))_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \widetilde{\mathrm{H}}_{n+1}(\mathrm{S}(\mathrm{S}^n);\mathbb{Z}) & \xrightarrow[\cong]{\delta} & \widetilde{\mathrm{H}}_n(\mathrm{S}^n;\mathbb{Z}). \end{array}$$

This also holds for $n = 0$: the reduced group is the kernel of the augmentation, and every map preserves this kernel, whether or not it preserves a chosen basepoint. Choose the suspended generators to correspond under δ. The square proves preservation of degree.

On $\mathrm{S}^0 = \{-1, 1\}$, interchange sends the generator $[1] - [-1]$ to its negative. Under the homeomorphism

$$\mathrm{S}(\mathrm{S}^{n-1}) \longrightarrow \mathrm{S}^n, \qquad [u, t] \longmapsto (\sin(\pi t)u, \cos(\pi t)),$$

reflection of the first coordinate is the suspension of the same reflection in one lower dimension. Induction therefore gives degree -1. Functoriality gives $\deg(gf) = \deg(g)\deg(f)$ and shows that conjugation by a homeomorphism preserves degree. Coordinate permutations conjugate the first reflection to each other coordinate reflection. Their composite is the antipodal map, so its degree is $(-1)^{n+1}$. In particular, this sign has been obtained on homology without an identification of degree with a determinant sign.

Let $\gamma(t) = \exp(2\pi i t)$ for $0 \leq t \leq 1$. This singular 1-simplex is a cycle. Its class generates $\mathrm{H}_1(\mathrm{S}^1;\mathbb{Z})$. Indeed, use the cover of Lecture 7, Example 7.2.9 with U the complement of $-i$ in the circle and V the complement of i. The upper and lower semicircle pieces lie in U and V, respectively. The connecting-map formula sends their sum to $[1] - [-1]$, a generator of the kernel in that example. To justify this splitting at the chain level, for any path $\eta : I \to X$ and $0 < a < 1$, the singular 2-simplex

$$(t_0, t_1, t_2) \longmapsto \eta(at_1 + t_2)$$

has boundary $\eta|_{[a,1]} - \eta + \eta|_{[0,a]}$, where the restricted paths are linearly reparametrized to I. Thus subdivision of a path into finitely many pieces changes its chain by a boundary.

For $m > 0$, subdivide $p_m\gamma$ at $0, 1/m, \dots, 1$. Each reparametrized piece is γ, so $(p_m)_*([\gamma]) = m[\gamma]$. The map p_0 is constant and has degree zero. Complex conjugation p_{-1} is a coordinate reflection of the circle, and $p_{-m} = p_{-1}p_m$ for $m > 0$, proving the formula for negative integers. Iterated suspension preserves the degree and the identities $p_a p_b = p_{ab}$, since suspension respects composition. These identities will give commuting attaching-map squares as well as the required degrees. □

A local contribution will mean an integer defined on relative homology, rather than a sign inferred from a picture. The next lemma proves that these integers add to the homological degree. For a finite subset E of S^n, write S^n_E for its complement. If $E = \{x\}$, write S^n_x.

Lemma 8.1.11. Let $f : \mathrm{S}^n \to \mathrm{S}^n$ with $n \geq 1$, and let y have finite inverse image $F = \{x_1, \dots, x_s\}$. Suppose f is a homeomorphism from a neighborhood of each x_i onto a neighborhood of y. Choose integral homological orientations of the source and target spheres. They induce generators of the local groups

$$\mathrm{H}_n(\mathrm{S}^n, \mathrm{S}^n_{x_i}; \mathbb{Z}), \qquad \mathrm{H}_n(\mathrm{S}^n, \mathrm{S}^n_y; \mathbb{Z}).$$

Let $\varepsilon_i \in \{1, -1\}$ be the coefficient of the local map induced by f in these generators. Then $\deg(f) = \sum_i \varepsilon_i$.

Proof. For any point $x \in \mathrm{S}^n$, stereographic projection identifies S^n_x with $\mathbb{R}^n$, so this complement is contractible. The reduced long exact sequence of its inclusion therefore makes the canonical map

$$j_x : \mathrm{H}_n(\mathrm{S}^n; \mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_n(\mathrm{S}^n, \mathrm{S}^n_x; \mathbb{Z})$$

an isomorphism, including when $n = 1$. The local generator is, by definition, the image of the global generator under j_x.

Choose pairwise disjoint open neighborhoods U_i of the x_i on which f is a homeomorphism onto a neighborhood of y. Put $U = \coprod_i U_i$ and $U_i^\times = U_i \cap \mathrm{S}^n_F$. The sets U and S^n_F form an open cover. Lecture 7, Corollary 7.2.5, followed by additivity, identifies

$$\begin{aligned} \mathrm{H}_n(\mathrm{S}^n, \mathrm{S}^n_F; \mathbb{Z}) &\cong \bigoplus_i \mathrm{H}_n(U_i, U_i^\times; \mathbb{Z}) \\ &\cong \bigoplus_i \mathrm{H}_n(\mathrm{S}^n, \mathrm{S}^n_{x_i}; \mathbb{Z}). \end{aligned}$$

The last isomorphisms are excision for the covers $U_i \cup (\mathrm{S}^n_{x_i})$. The map from the first relative group to the relative group at x_i is the projection onto the ith summand: every other summand is represented inside $\mathrm{S}^n_{x_i}$ and hence maps to zero. Consequently the image of the global generator has coefficient 1 in every summand.

The identity $f^{-1}(y) = F$ gives a map of subspace inclusions from $\mathrm{S}^n_F \hookrightarrow \mathrm{S}^n$ to $\mathrm{S}^n_y \hookrightarrow \mathrm{S}^n$. On the ith excision summand its relative homology map is the local map of f. This map is an isomorphism because the restriction is a homeomorphism and both local identifications are excision isomorphisms; thus its coefficient is $\varepsilon_i \in \{1, -1\}$. Naturality now gives, in the chosen generators, the diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1 \mapsto (1,\dots,1)} & \mathbb{Z}^s \\ {\scriptstyle \times \deg(f)}\downarrow & & \downarrow{\scriptstyle (a_i) \mapsto \sum_i \varepsilon_i a_i} \\ \mathbb{Z} & = & \mathbb{Z}. \end{array}$$

Its commutativity proves the formula. If F is empty, the map factors through the contractible complement of y, so its degree is zero, the empty sum. □

The local-degree formula requires a finite inverse image and local homeomorphisms. Counting inverse images without their signs does not in general compute degree. The local maps are obtained on small neighborhoods by excision, not from maps of the whole sphere punctured at a single inverse image.

For example, consider $p_m : \mathrm{S}^1 \to \mathrm{S}^1$, $p_m(z) = z^m$, for an integer $m > 0$. Every target point has m inverse images. In angular coordinates the local map is $t \mapsto mt$. On a small arc this map extends to an orientation-preserving piecewise linear homeomorphism of the circle. Such a homeomorphism is homotopic to the identity by linearly interpolating an increasing lift to $\mathbb{R}$ with the identity lift. Its global degree, and hence its local degree, is 1. Each contribution is therefore 1, and $\deg(p_m) = m$. Complex conjugation has degree -1, and composition gives $\deg(p_m) = m$ also for negative m. For $m = 0$ the map is constant.

8.2 Cellular Homology and Products

We use ordinary singular homology with coefficients in an abelian group A: in addition to homotopy invariance, excision, exactness, and additivity, the dimension axiom says that $\mathrm{H}_0(*; A) = A$ and $\mathrm{H}_q(*; A) = 0$ for $q \neq 0$. In the computations below, cell complexes are finite, as in Definition 8.1.2. Their n-skeleta, denoted $\mathrm{sk}_n X$, consist of their cells of dimension at most n; put $\mathrm{sk}_r X = \emptyset$ for $r < 0$.

8.2.1 Relative Cells and Cellular Homology

For $n \geq 0$, the sphere calculation of Lecture 5, Proposition 5.3.6, gives

$$\widetilde{\mathrm{H}}_q(\mathrm{S}^n; A) = \begin{cases} A, & q = n, \\ 0, & q \neq n. \end{cases}$$

Here reduced homology has the convention of Lecture 5, Definition 5.3.1, including S^0. For $n > 0$, ordinary homology also has its copy of A in degree zero.

To apply open-cover excision to the characteristic disks, fix $n > 0$ and write $Y = \mathrm{sk}_n X$ and $Y' = \mathrm{sk}_{n-1} X$. Let D be the disjoint union of the characteristic disks, let $E \subset D$ be their boundaries, and let $j : D \to Y$ be the characteristic map. The attaching construction is the pushout

$$\begin{array}{ccc} E & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ D & \xrightarrow[j]{} & Y. \end{array}$$

Let $N \subset D$ be the union of the collars $\|x\| > 1/2$, including the boundaries, and put $U = Y' \cup j(N)$. Let V_D be the union of the disk interiors and $V = j(V_D)$ the union of the open n-cells. The quotient topology makes U, V an open cover of Y: their inverse images in $Y' + D$ are $Y' + N$ and V_D. Likewise, N, V_D cover D by open sets. On the interiors, j gives a homeomorphism of pairs from $(V_D, N \cap V_D)$ to $(V, U \cap V)$, regardless of any boundary identifications.

Radial motion $r \mapsto (1-t)r + t$ retracts N onto E. With Y' fixed, it also retracts U onto Y'. These homotopies are strong deformation retractions: they fix the boundaries, descend through the attaching quotient, and are continuous by the product-quotient lemma of Lecture 7, Lemma 7.1.7, applied to the open saturated inverse image $Y' + N$. The relative long exact sequences

and open-cover excision, Lecture 7, Corollary 7.2.5, now give the commutative diagram

$$\begin{array}{ccc}
\mathrm{H}_q(D,E;A) & \xrightarrow{\;j_*\;} & \mathrm{H}_q(Y,Y';A) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{H}_q(D,N;A) & \xrightarrow[\;j_*\;]{} & \mathrm{H}_q(Y,U;A) \\
\cong\uparrow & & \uparrow\cong \\
\mathrm{H}_q(V_D,N\cap V_D;A) & \xrightarrow[\cong]{} & \mathrm{H}_q(V,U\cap V;A).
\end{array}$$

The lower vertical maps are excision, and the bottom map is the homeomorphism of pairs. Hence the top map is an isomorphism. Additivity for the disjoint disks identifies the characteristic-map homomorphism

$$\bigoplus_{n\text{-cells of } X} \mathrm{H}_q(\mathrm{D}^n,\mathrm{S}^{n-1};A) \xrightarrow{\;\cong\;} \mathrm{H}_q(\mathrm{sk}_n X,\mathrm{sk}_{n-1} X;A).$$

For $n=0$, this is the direct-sum decomposition for a finite set of points, using $(\mathrm{D}^0,\emptyset)$.

Choose an orientation of each cell. The disk-pair long exact sequence and the sphere calculation show that these groups vanish unless $q=n$, when each cell contributes one copy of A. A degree-d map of an oriented sphere acts by multiplication by d with coefficients in A: the coefficient maps $\mathbb{Z}\to A$, sending 1 to each element of A, carry the integral degree formula to this formula by naturality.

Definition 8.2.1. The *cellular chain complex* of X with coefficients in A has chain groups

$$\mathrm{C}_n^{\mathrm{cell}}(X;A)=\mathrm{H}_n(\mathrm{sk}_n X,\mathrm{sk}_{n-1} X;A).$$

Its differential is the composite in the diagram

$$\mathrm{H}_n(\mathrm{sk}_n X,\mathrm{sk}_{n-1} X;A) \xrightarrow{\;\delta\;} \mathrm{H}_{n-1}(\mathrm{sk}_{n-1} X;A) \longrightarrow \mathrm{H}_{n-1}(\mathrm{sk}_{n-1} X,\mathrm{sk}_{n-2} X;A).$$

Write $j_r:\mathrm{H}_r(\mathrm{sk}_r X;A)\to\mathrm{H}_r(\mathrm{sk}_r X,\mathrm{sk}_{r-1} X;A)$ for the relative comparison. Then $\partial_n=j_{n-1}\delta_n$, and $\delta_{n-1}j_{n-1}=0$ by exactness for the pair $(\mathrm{sk}_{n-1} X,\mathrm{sk}_{n-2} X)$. Thus $\partial_{n-1}\partial_n=0$; put $\partial_0=0$.

Here is the attaching-coefficient calculation for $n\geq 2$. Fix an oriented n-cell with characteristic map $\chi:(\mathrm{D}^n,\mathrm{S}^{n-1})\to(\mathrm{sk}_n X,\mathrm{sk}_{n-1} X)$ and attaching map $a=\chi|_{\mathrm{S}^{n-1}}$. Its relative generator ω has boundary the chosen sphere generator ζ. Naturality gives $\delta_n(\chi_*(\omega))=a_*(\zeta)$. For an $(n-1)$-cell e, collapse the complement of its open cell in $\mathrm{sk}_{n-1} X$ to a point. The resulting quotient is its characteristic disk with boundary collapsed, hence an oriented S^{n-1}; denote the collapse by q_e. It gives a map of pairs

$$q_e:(\mathrm{sk}_{n-1} X,\mathrm{sk}_{n-2} X)\longrightarrow(\mathrm{S}^{n-1},*).$$

On the characteristic relative classes this map kills every other cell and sends the class of e to the chosen generator: on the latter disk it is exactly the disk-boundary collapse. This last assertion follows also by applying the earlier collar-excision argument to the disk and its collapsed boundary. Thus $(q_e)_*$ is projection onto the e summand. The map $\mathrm{H}_{n-1}(\mathrm{S}^{n-1};\mathbb{Z})\to\mathrm{H}_{n-1}(\mathrm{S}^{n-1},*;\mathbb{Z})$ is an isomorphism, including $n=2$, by the reduced relative exact sequence. Applying $(q_e)_*j_{n-1}$ to $a_*(\zeta)$ shows that the coefficient is precisely $\deg(q_e\circ a)$. For $n=1$, use the interval boundary directly: its terminal vertex minus its initial vertex, with cancellation if

they coincide. No degree of a map of 0-spheres is needed. Naturality with respect to $\mathbb{Z} \to A$, $1 \mapsto a$ for each $a \in A$, now gives the same integer incidence coefficients with coefficients in A. Therefore

$$\mathrm{C}_\bullet^{\mathrm{cell}}(X; A) = \mathrm{C}_\bullet^{\mathrm{cell}}(X; \mathbb{Z}) \otimes_{\mathbb{Z}} A.$$

Theorem 8.2.2. For every finite cell complex X and abelian group A, there are isomorphisms $\mathrm{H}_n(\mathrm{C}_\bullet^{\mathrm{cell}}(X; A)) \cong \mathrm{H}_n(X; A)$.

Proof. For brevity write $X_r = \mathrm{sk}_r X$ and $C_r = \mathrm{C}_r^{\mathrm{cell}}(X; A)$. The relative-cell calculation gives $\mathrm{H}_q(X_r, X_{r-1}; A) = 0$ for $q \neq r$. Starting with X_0, a finite set of points, induction in the exact sequence

$$\mathrm{H}_q(X_{r-1}; A) \longrightarrow \mathrm{H}_q(X_r; A) \longrightarrow \mathrm{H}_q(X_r, X_{r-1}; A)$$

shows $\mathrm{H}_q(X_r; A) = 0$ for $q > r$.

Fix $n \geq 1$. The following exact segments display all maps needed for the comparison:

$$\begin{aligned} 0 = \mathrm{H}_n(X_{n-1}; A) \longrightarrow \mathrm{H}_n(X_n; A) &\xrightarrow{j_n} C_n \xrightarrow{\delta_n} \mathrm{H}_{n-1}(X_{n-1}; A), \\ 0 = \mathrm{H}_{n-1}(X_{n-2}; A) \longrightarrow \mathrm{H}_{n-1}(X_{n-1}; A) &\xrightarrow{j_{n-1}} C_{n-1}, \\ C_{n+1} \xrightarrow{\delta_{n+1}} \mathrm{H}_n(X_n; A) \longrightarrow \mathrm{H}_n(X_{n+1}; A) &\longrightarrow \mathrm{H}_n(X_{n+1}, X_n; A) = 0. \end{aligned}$$

The first two lines make j_n, j_{n-1} injective and give $\ker \partial_n = \ker(j_{n-1}\delta_n) = \ker \delta_n = \operatorname{im} j_n$. Write $p : \mathrm{H}_n(X_n; A) \to \mathrm{H}_n(X_{n+1}; A)$ for the inclusion-induced map. The relevant diagram has exact rows:

$$\begin{array}{ccccccc} C_{n+1} & \xrightarrow{\delta_{n+1}} & \mathrm{H}_n(X_n; A) & \xrightarrow{p} & \mathrm{H}_n(X_{n+1}; A) & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle j_n} & & \uparrow{\scriptstyle \theta} & & \\ C_{n+1} & \xrightarrow[\partial_{n+1}]{} & \ker \partial_n & \xrightarrow[\pi]{} & \ker \partial_n / \operatorname{im} \partial_{n+1} & \longrightarrow & 0. \end{array}$$

Here j_n is regarded as a map onto its image $\ker \partial_n$, and the left square commutes because $j_n \delta_{n+1} = \partial_{n+1}$. Define θ by the following chase.

- For $c \in \ker \partial_n$, there is a unique $a \in \mathrm{H}_n(X_n; A)$ with $j_n(a) = c$. Set $\theta([c]) = p(a)$.
- If c is replaced by $c + \partial_{n+1} b$, its unique lift is $a + \delta_{n+1} b$. Since $p\delta_{n+1} = 0$, the value of θ is unchanged. Thus θ is a well-defined homomorphism.
- Given $z \in \mathrm{H}_n(X_{n+1}; A)$, exactness gives a with $p(a) = z$; then $\theta([j_n(a)]) = z$. Thus θ is surjective.
- If $\theta([c]) = 0$, lift $c = j_n(a)$. Exactness gives $a = \delta_{n+1} b$, hence $c = j_n \delta_{n+1} b = \partial_{n+1} b$ and $[c] = 0$. Thus θ is injective.

Consequently

$$\ker \partial_n / \operatorname{im} \partial_{n+1} \cong \mathrm{H}_n(X_{n+1}; A).$$

For $n = 0$, $C_0 = \mathrm{H}_0(X_0; A)$ and $\partial_0 = 0$. The third exact segment, with $n = 0$, gives the same quotient conclusion directly.

Finally, for $r \geq n + 2$, both outside terms in

$$\mathrm{H}_{n+1}(X_r, X_{r-1}; A) \longrightarrow \mathrm{H}_n(X_{r-1}; A) \longrightarrow \mathrm{H}_n(X_r; A) \longrightarrow \mathrm{H}_n(X_r, X_{r-1}; A)$$

vanish. The middle map is therefore an isomorphism. There are finitely many skeleta before reaching X, so $\mathrm{H}_n(X_{n+1}; A) \cong \mathrm{H}_n(X; A)$. This proves the theorem in every nonnegative degree; both sides vanish in negative degrees. □

The dimension axiom is used in the vanishing of the relative-cell groups outside one degree. The same argument does not give this cellular complex for a generalized homology theory with nonzero coefficient groups in other degrees.

8.2.2 Product Cells

For ordinary homology, products of cells identify the chain complex of a product with a tensor product. We first construct the canonical homology cross product and then give hypotheses under which it is an isomorphism. A geometric treatment of product cells and their tensor-product signs is given in [May99, Chapter 13, §4, pp. 101–103].

A cell of $X \times Y$ is the product of a cell of X and a cell of Y, oriented in that order. Put $\mathrm{S}^{-1} = \emptyset$ when a factor is a 0-cell. The oriented boundary of $\mathrm{D}^p \times \mathrm{D}^q$ is

$$\partial(\mathrm{D}^p \times \mathrm{D}^q) = (\mathrm{S}^{p-1} \times \mathrm{D}^q) \cup (\mathrm{D}^p \times \mathrm{S}^{q-1}).$$

The two faces are glued along $\mathrm{S}^{p-1} \times \mathrm{S}^{q-1}$, and the product characteristic map sends their union into $\mathrm{sk}_{p+q-1}(X \times Y)$. Give the product disk the orientation specified by the signed shuffle triangulation below. The boundary calculation there gives sign $+1$ on the first kind of face and $(-1)^p$ on the second kind. Consequently, identifying oriented product cells with tensors gives an isomorphism

$$\mathrm{C}_\bullet^{\mathrm{cell}}(X \times Y; \mathbb{Z}) \cong \mathrm{C}_\bullet^{\mathrm{cell}}(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{C}_\bullet^{\mathrm{cell}}(Y; \mathbb{Z}), \qquad \partial(\lambda \otimes \mu) = \partial\lambda \otimes \mu + (-1)^p \lambda \otimes \partial\mu \quad (\lambda \in \mathrm{C}_p^{\mathrm{cell}}(X; \mathbb{Z})).$$

Indeed, restrict each product characteristic map to these two boundary pieces and use naturality of the relative connecting homomorphism. The relative product boundary identity proved below gives the original incidence coefficients on the first piece and $(-1)^p$ times those on the second. Their common boundary maps into the $(p+q-2)$-skeleton, so contributes zero to the relative group defining the differential. These account for every $(p+q-1)$-cell. The product disk is a disk of dimension $p+q$: for positive-dimensional factors, radial rescaling from the Euclidean unit ball by the continuous gauge $(x, y) \mapsto \max(\|x\|, \|y\|)$ gives a boundary-preserving homeomorphism. A zero-dimensional factor changes nothing.

Let R be a commutative ring. Triangulate $|\Delta^p| \times |\Delta^q|$ by monotone lattice paths from $(0, 0)$ to (p, q). A path with successive lattice vertices (i_k, j_k), $0 \le k \le p + q$, determines the ordered affine simplex with vertices (e_{i_k}, e_{j_k}). Give it coefficient $(-1)^N$, where N counts pairs consisting of a vertical step followed later by a horizontal step. These simplices form a triangulation: the cumulative coordinates of a point in the two factors can be interleaved in nondecreasing order, and the resulting order determines its path simplex; ties give their common faces. Thus their interiors are disjoint and their union is the product. For singular simplices σ and τ, composing this signed triangulation with $\sigma \times \tau$ and extending R-bilinearly defines the natural chain map

$$\boxtimes : \mathrm{C}_\bullet(X; R) \otimes_R \mathrm{C}_\bullet(Y; R) \longrightarrow \mathrm{C}_\bullet(X \times Y; R).$$

Here is the boundary calculation, including the signs. Deleting the kth vertex of an ordered path simplex contributes $(-1)^k$. If the two steps adjacent to that vertex have different directions, swapping those steps gives the same ordered deleted-vertex face in a second path. The two values of N differ by one, so these interior faces cancel as identical parametrized simplices.

The remaining faces lie on a face of one factor. For two horizontal steps adjacent to a vertex (i, j), deleting that vertex gives the ith face of the first factor. Since $k = i + j$, and removing the redundant horizontal step changes N by j, its coefficient relative to the smaller shuffle is $(-1)^{i+j}(-1)^j = (-1)^i$. For two vertical steps, the analogous change in N is $p - i$, and the sign is $(-1)^{i+j}(-1)^{p-i} = (-1)^{p+j}$, the jth face sign of the second factor with the extra $(-1)^p$. Deleting

the initial or terminal vertex satisfies the same formulas, according to the direction of the initial or terminal step. These cases exhaust the faces, proving

$$\partial(\sigma \boxtimes \tau) = \partial\sigma \boxtimes \tau + (-1)^p \sigma \boxtimes \partial\tau.$$

For a zero-dimensional factor this identity follows directly from the single path and the zero differential in degree zero.

The same formula passes to relative chains: chains in either specified subspace map into its product with the other factor. On oriented disks the signed triangulation defines the product relative generator. One can check it is a generator by restricting to a point in the interior of a top-dimensional triangulation simplex: all other simplices disappear in local relative homology, and its coefficient is $+1$ or -1. Excision identifies the local group with that of the interior simplex; radial retraction of the punctured disk onto its boundary and the relative exact sequences identify it with the disk's relative top homology. The displayed boundary calculation therefore proves the relative product boundary identity, and hence the product-cell signs used above. Thus products of cycles define a canonical homomorphism of graded R-modules

$$\boxtimes : \mathrm{H}_*(X;R) \otimes_R \mathrm{H}_*(Y;R) \longrightarrow \mathrm{H}_*(X \times Y;R), \qquad (M_* \otimes_R N_*)_n = \bigoplus_{p+q=n} M_p \otimes_R N_q.$$

For the interchange map $t : X \times Y \to Y \times X$, exchanging horizontal and vertical steps replaces N by $pq - N$. Thus every corresponding simplex changes sign by $(-1)^{pq}$, proving $t_*(\alpha \boxtimes \beta) = (-1)^{pq}\beta \boxtimes \alpha$ in degrees p, q. On oriented relative cells, this construction gives the product orientation used above.

Theorem 8.2.3. Let R be a principal ideal domain and let X, Y be finite cell complexes whose homology groups over R are free. The canonical cross product is an isomorphism

$$\mathrm{H}_*(X;R) \otimes_R \mathrm{H}_*(Y;R) \xrightarrow{\cong} \mathrm{H}_*(X \times Y;R).$$

In particular, this holds over any field. For finite cell complexes $X_1, \dots, X_r$ with free integral homology, the graded tensor product over $\mathbb{Z}$ gives

$$\mathrm{H}_*\left(\prod_{i=1}^{r} X_i;\mathbb{Z}\right) \cong \bigotimes_{i=1}^{r} \mathrm{H}_*(X_i;\mathbb{Z}).$$

For coefficients in an abelian group A, tensor this graded group over $\mathbb{Z}$ with A. Its degree-k part is, with all tensor products over $\mathbb{Z}$,

$$\mathrm{H}_k\left(\prod_{i=1}^{r} X_i;A\right) \cong \bigoplus_{k_1+\cdots+k_r=k} \left(\bigotimes_{i=1}^{r} \mathrm{H}_{k_i}(X_i;\mathbb{Z})\right) \otimes_{\mathbb{Z}} A.$$

Proof. Over a principal ideal domain, a finite free chain complex with free homology splits into its homology, with zero differential, and contractible two-term complexes. To see this, first split cycles onto homology; then split each chain group onto the boundaries in the next degree, which are free by Lecture 3, Theorem 3.1.10. Tensoring preserves these contracting homotopies. The relative cross product sends oriented cell generators to the oriented product-cell generator. A cellular cycle of degree p corresponds to a class in $\mathrm{H}_p(\mathrm{sk}_p X;R)$. Naturality for the skeletal pairs therefore identifies products of cellular cycles with the singular cross product under Theorem 8.2.2. Thus the induced homology map is the displayed canonical map. Apply the product-cell isomorphism and the splittings to obtain the isomorphism. For the integral statement, iterate for finitely many factors and then tensor with A. No flatness assumption on A is needed. □

8.2.3 Even Cells and Products of Spheres

Definition 8.2.4. A finite cell complex is *even-cell* if every cell has even dimension.

Its cellular differential is zero. Products of even-cell complexes are again even-cell, because dimensions add. Thus if c_{2r} is the number of $2r$-cells, then

$$\mathrm{H}_{2r}(X;A) \cong A^{\oplus c_{2r}}, \qquad \mathrm{H}_{2r+1}(X;A) = 0.$$

This is a condition on cells. If instead one only knows that integral homology is even and free, Theorem 8.2.3 gives the corresponding product calculation.

For $n_1, \dots, n_r > 0$, give each sphere S^{n_i} its one 0-cell and one n_i-cell. Each differential is zero, so product cells give

$$\mathrm{H}_k\left(\prod_{i=1}^{r} \mathrm{S}^{n_i}; A\right) \cong \bigoplus_{\substack{J\subseteq\{1,\dots,r\} \\ \sum_{j\in J} n_j = k}} A.$$

The special case of products of circles gives the torus calculation.

Theorem 8.2.5. For $r \geq 0$ and every abelian group A, the torus $T^r = (\mathrm{S}^1)^r$ has homology

$$\mathrm{H}_k(T^r;A) \cong A^{\oplus \binom{r}{k}} \qquad (0 \leq k \leq r),$$

and zero homology in other degrees. Over $\mathbb{Z}$, the isomorphism is the iterated graded cross product of the circle homology groups.

Proof. The integral homology of a circle is $\mathbb{Z}$ in degrees zero and one and vanishes otherwise. By Theorem 8.2.3, the degree-k part of its r-fold graded tensor product has one generator for every choice of k factors contributing degree one. There are $\binom{r}{k}$ such choices. Tensoring with A gives the claim. Equivalently, the two endpoints of the 1-cell of a circle attach to the same vertex with opposite signs; the product boundary formula gives zero differential in every torus cell. □

8.3 Projective Spaces and Mapping Tori

8.3.1 Complex Projective Spaces

Definition 8.3.1. *Complex projective space* CP^n is the space of complex lines in $\mathbb{C}^{n+1}$.

The standard inclusions give a filtration $\mathrm{CP}^0 \subseteq \mathrm{CP}^1 \subseteq \cdots \subseteq \mathrm{CP}^n$.

Theorem 8.3.2. For $n \geq 0$ and an abelian group A,

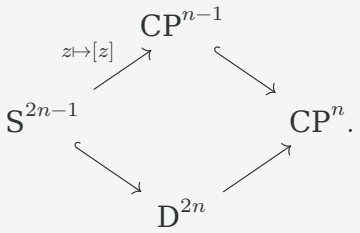
$$\mathrm{H}_k(\mathrm{CP}^n;A) \cong \begin{cases} A, & k = 0, 2, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

For $n \geq 1$, the standard inclusion $\mathrm{CP}^{n-1} \to \mathrm{CP}^n$ induces isomorphisms in degrees below $2n$.

Proof. For $n > 0$, the lines with last coordinate nonzero form an affine chart $\mathbb{C}^n$. A characteristic map for this cell is

$$\mathrm{D}^{2n} \longrightarrow \mathrm{CP}^n, \qquad z \longmapsto [z : \sqrt{1 - \|z\|^2}],$$

where $z \in \mathbb{C}^n$ and $\|z\| \le 1$. On the boundary it is $z \mapsto [z : 0]$, so it gives the pushout



On the boundary $\|z\| = 1$, this map is $z \mapsto [z : 0] \in \mathrm{CP}^{n-1}$. On the interior it identifies with the homeomorphism $z \mapsto z/\sqrt{1 - \|z\|^2}$ onto the affine chart $\mathbb{C}^n$; its inverse is $w \mapsto w/\sqrt{1 + \|w\|^2}$. The induced map from the displayed pushout to CP^n is therefore bijective: its interior points give the chart, and its boundary identifications give exactly CP^{n-1}. The pushout is compact, and CP^n is Hausdorff, so this continuous bijection is a homeomorphism. This proves the pushout assertion. Since the sphere inclusion into the disk is a cofibration, the square is also a homotopy pushout. Starting with $\mathrm{CP}^0 = *$ gives one cell in each dimension $0, 2, \dots, 2n$. Its cellular differential is zero. The cellular theorem proves the formula and the assertion about inclusions. □

For example, $\mathrm{CP}^m \times \mathrm{CP}^n$ has one copy of A in degree $2k$ for each pair (i, j) with $0 \le i \le m$, $0 \le j \le n$, and $i + j = k$; its odd homology vanishes. For a chosen finite cell structure on X, its *cell-count polynomial* is $\sum_k c_k t^k$, where c_k is the number of k-cells. Product-cell dimensions add, so multiplication of these polynomials counts product cells: the coefficient of t^k is $\sum_{i+j=k} c_i(X) c_j(Y)$. Thus the cell-count polynomial of a finite product of complex projective spaces is

$$\prod_{i=1}^{r} (1 + t^2 + \cdots + t^{2n_i}) \quad \text{for} \quad \prod_{i=1}^{r} \mathrm{CP}^{n_i}.$$

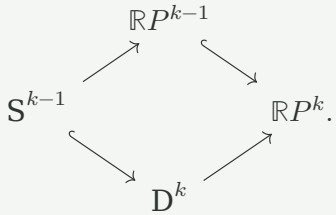
Each factor records the one cell in every even dimension of the corresponding projective space. In general a cell-count polynomial depends on the chosen cell structure and is not the homology-rank polynomial. Here every cellular differential is zero, so its coefficients also give the integral Betti numbers. The coefficient of t^k counts the copies of A in degree k.

8.3.2 Real Projective Spaces

Definition 8.3.3. *Real projective space* $\mathbb{R}P^n$ is the space of real lines in $\mathbb{R}^{n+1}$.

Its standard filtration has one cell in every dimension from 0 through n; unlike the complex case, adjacent cell dimensions occur.

The upper hemisphere of S^k gives a characteristic disk for the k-cell. On its boundary, antipodal points are identified, giving the pushout



For $k \ge 2$, let $q_{k-1} : \mathrm{S}^{k-1} \to \mathbb{R}P^{k-1}$ send a unit vector to its line, and let $c : \mathbb{R}P^{k-1} \to \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2}$ be the collapse map. The characteristic disk identifies this quotient with $\mathrm{D}^{k-1}/\mathrm{S}^{k-2} \cong \mathrm{S}^{k-1}$. Write $f_k = c q_{k-1}$ for the map whose degree is the cellular coefficient.

The two open hemispheres $U_+ = \{u_k > 0\}$ and $U_- = \{u_k < 0\}$ in $\mathrm{S}^{k-1} \subseteq \mathbb{R}^k$ map homeomorphically under f_k onto the complement V of the collapsed point. Indeed, a line outside $\mathbb{R}P^{k-2}$ has a unique unit representative with last coordinate positive, and a unique one with last coordinate negative; these representatives depend continuously on the line. Choose y in that open cell, and write its inverse image as $\{x, -x\}$ with $x \in U_+$. The local-degree formula of Lemma 8.1.11, gives

$$\deg(f_k) = \varepsilon_+ + \varepsilon_-.$$

Let $U_+^\times$, $U_-^\times$, and $V^\times$ be U_+, U_-, and V with x, $-x$, and y, respectively, removed. Let $a(u) = -u$. Since $q_{k-1}a = q_{k-1}$, the two local maps of f_k satisfy the commutative square

$$\begin{array}{ccc} \mathrm{H}_{k-1}(U_+, U_+^\times; \mathbb{Z}) & \xrightarrow[\cong]{a_*} & \mathrm{H}_{k-1}(U_-, U_-^\times; \mathbb{Z}) \\ \downarrow{\scriptstyle (f_k|_{U_+})_*} & & \downarrow{\scriptstyle (f_k|_{U_-})_*} \\ \mathrm{H}_{k-1}(V, V^\times; \mathbb{Z}) & = & \mathrm{H}_{k-1}(V, V^\times; \mathbb{Z}). \end{array}$$

The local generators are the images of the global generator, as in the proof of that lemma. Naturality of these images shows that a_* multiplies them by the global degree of a, which is $(-1)^k$ by Lemma 8.1.10. The square therefore gives $\varepsilon_+ = (-1)^k \varepsilon_-$. Choose the orientation of the k-cell so that $\varepsilon_+ = 1$. It follows that $\varepsilon_- = (-1)^k$ and hence $\deg(f_k) = 1 + (-1)^k$.

For $k = 1$, the two endpoints of the characteristic interval attach to the same vertex. The relative boundary of the interval is its terminal endpoint minus its initial endpoint, so their images cancel. The coefficient formula holds over every abelian group A by naturality in coefficients. Therefore

$$\partial_k = 1 + (-1)^k = \begin{cases} 2, & k \text{ even}, \\ 0, & k \text{ odd}, \end{cases} \qquad (1 \le k \le n),$$

$$\mathrm{C}_k^{\mathrm{cell}}(\mathbb{R}P^n; A) = A \quad (0 \le k \le n).$$

Set $\partial_0 = 0$. The orientation of each even-dimensional cell may be chosen to give $+2$; this does not change neighboring nonzero differentials, since the odd differentials vanish.

Theorem 8.3.4. The cellular differential of real projective space alternates between zero and multiplication by two. Write $A[2] = \{a \in A : 2a = 0\}$. For $n \ge 1$, the resulting homology is

$$\mathrm{H}_k(\mathbb{R}P^n; A) \cong \begin{cases} A, & k = 0, \\ A/2A, & 0 < k < n \text{ and } k \text{ odd}, \\ A[2], & 0 < k < n \text{ and } k \text{ even}, \\ A, & k = n \text{ and } n \text{ odd}, \\ A[2], & k = n \text{ and } n \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

For $n = 0$, projective space is a point.

Proof. The attaching-degree calculation gives the cellular complex. In an odd degree below the top, its homology is $A/2A$; in a positive even degree it is $\ker(2 : A \to A)$. At the top there is no incoming differential. The cellular theorem gives all the stated groups. □

With integral coefficients, the positive groups below the top are $\mathbb{Z}/2\mathbb{Z}$ in odd degrees and zero in even degrees. The top group is $\mathbb{Z}$ for odd n and zero for positive even n. With coefficients in $\mathbb{Z}/2\mathbb{Z}$, every differential vanishes, so every degree from 0 to n contains one copy of $\mathbb{Z}/2\mathbb{Z}$.

For instance, $\mathbb{R}P^2$ has integral groups $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, and 0 in degrees $0,1,2$, while its mod-two homology is nonzero in all three degrees. These are computations of groups; a cohomology ring calculation additionally requires a cup product.

8.3.3 Mapping Tori

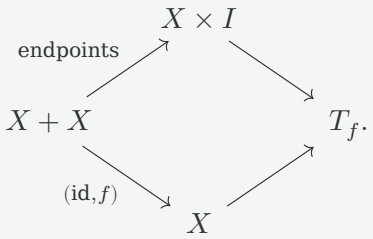
A final gluing calculation distinguishes the torus from the Klein bottle.

Definition 8.3.5. For a continuous map $f : X \to X$, let $i_0, i_1 : X \to X \times I$ be the endpoint inclusions. Its *mapping torus* is the coequalizer in Top

$$X \underset{i_0 \circ f}{\overset{i_1}{\rightrightarrows}} X \times I \xrightarrow{\;q\;} T_f.$$

Thus $qi_1 = qi_0 f$, and every continuous map $g : X \times I \to Y$ satisfying $gi_1 = gi_0 f$ factors uniquely as $g = \bar{g}q$ with $\bar{g} : T_f \to Y$ continuous. Concretely, q identifies $(x, 1)$ with $(f(x), 0)$, and T_f carries the quotient topology.

It is the pushout



The endpoint inclusion is a cofibration, so Lecture 7, Proposition 7.1.6, makes this a homotopy pushout. After retracting $X \times I$ onto X, its Mayer–Vietoris map has matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & f_* \end{pmatrix}.$$

Subtract the first row from the second and then the first column from the second. This separates an identity summand from $f_* - 1$. Cancelling the identity summand in the exact sequence, and changing the sign of the remaining summand, gives the *Wang sequence*

$$\cdots \longrightarrow \mathrm{H}_k(X; A) \xrightarrow{1 - f_*} \mathrm{H}_k(X; A) \longrightarrow \mathrm{H}_k(T_f; A) \longrightarrow \mathrm{H}_{k-1}(X; A) \xrightarrow{1 - f_*} \mathrm{H}_{k-1}(X; A) \longrightarrow \cdots.$$

If $X = \mathrm{S}^1$ and f has degree d, this gives

$$\mathrm{H}_1(T_f; A) \cong A \oplus A/(d-1)A, \qquad \mathrm{H}_2(T_f; A) \cong \{a \in A : (d-1)a = 0\},$$

with $\mathrm{H}_0(T_f; A) = A$ and higher groups zero. Naturality identifies the quotient map in degree one with the map induced by $T_f \to \mathrm{S}^1$. Choose $x \in \mathrm{S}^1$ and a path γ from $f(x)$ to x. The map $t \mapsto [\gamma(t), t]$ defines a section, since its endpoints are identified in T_f. It therefore splits the sequence in degree one.

For the identity map, T_f is the torus. For reflection of the circle, it is the Klein bottle, and $d = -1$ gives

$$\mathrm{H}_1(T_f; A) \cong A \oplus A/2A, \qquad \mathrm{H}_2(T_f; A) \cong A[2].$$

In particular, the Klein bottle has integral groups $\mathrm{H}_1(T_f; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{H}_2(T_f; \mathbb{Z}) = 0$, but its mod-two top homology is $\mathbb{Z}/2\mathbb{Z}$.

Mapping Complexes and Chain Homotopies

Algebraic Topology I | Fall 2026

A.1 Mapping Complexes and Chain Homotopies

The tensor product and mapping complex express chain homotopies through an interval complex. We use the homological grading of Lecture 2. Write Ab for the category of abelian groups and Ch(Ab) for its category of chain complexes. For a complex $A_\bullet$, write ∂_A for its differential.

A.1.1 The Tensor–Mor Adjunction

The mapping complex is the homological form of Lecture 3, Remark 3.3.5, obtained by reindexing cochains. It retains graded maps as well as chain maps. Its differential records which graded maps commute with the differentials of the two complexes.

Definition A.1.1. For complexes $A_\bullet, B_\bullet$ of abelian groups, define the tensor product by

$$(A \otimes B)_n = \bigoplus_{p+q=n} A_p \otimes_{\mathbb{Z}} B_q, \qquad \partial(\lambda \otimes \mu) = \partial_A \lambda \otimes \mu + (-1)^p \lambda \otimes \partial_B \mu \quad (\lambda \in A_p).$$

Define the *mapping complex* by

$$\underline{\mathrm{Mor}}_{\mathrm{Ch(Ab)}}(A, B)_r = \prod_{n \in \mathbb{Z}} \mathrm{Mor}_{\mathrm{Ab}}(A_n, B_{n+r}), \qquad \partial\phi = \partial_B \phi - (-1)^r \phi \partial_A.$$

Thus ∂ lowers degree by one. Its degree-zero cycles are chain maps, and a degree-one element s with $\partial s = f_1 - f_0$ is a chain homotopy from f_0 to f_1.

For three complexes $A_\bullet, B_\bullet, C_\bullet$, composition of graded maps gives a chain map

$$\underline{\mathrm{Mor}}_{\mathrm{Ch(Ab)}}(B, C) \otimes \underline{\mathrm{Mor}}_{\mathrm{Ch(Ab)}}(A, B) \longrightarrow \underline{\mathrm{Mor}}_{\mathrm{Ch(Ab)}}(A, C).$$

Indeed, for ψ of degree s, $\partial(\psi\phi) = (\partial\psi)\phi + (-1)^s \psi(\partial\phi)$. Composition is associative, and the identity maps are degree-zero cycles. Hence the morphisms between complexes themselves form complexes, compatibly with composition.

Proposition A.1.2. For complexes of abelian groups, there is a natural isomorphism of chain complexes

$$\underline{\mathrm{Mor}}_{\mathrm{Ch(Ab)}}(A \otimes B, C) \cong \underline{\mathrm{Mor}}_{\mathrm{Ch(Ab)}}(A, \underline{\mathrm{Mor}}_{\mathrm{Ch(Ab)}}(B, C)).$$

It sends a homogeneous map F to the map $\widetilde{F}$ defined by $\widetilde{F}(\lambda)(\mu) = F(\lambda \otimes \mu)$.

Proof. Degreewise, the isomorphism is the tensor–Mor adjunction for abelian groups, with the direct sums in the tensor product becoming products of morphism groups. For F of degree r and $\lambda \in A_p$, the two differentials agree since

$$\begin{aligned} \widetilde{\partial F}(\lambda)(\mu) &= \partial_C F(\lambda \otimes \mu) - (-1)^r F(\partial_A \lambda \otimes \mu) \\ &\qquad - (-1)^{r+p} F(\lambda \otimes \partial_B \mu) \\ &= (\partial \widetilde{F})(\lambda)(\mu). \end{aligned}$$

□

For fixed B, this says concretely that forming $A \otimes B$ is left adjoint to forming $\underline{\mathrm{Mor}}_{\mathrm{Ch(Ab)}}(B, C)$, including at the level of mapping complexes.

A.1.2 The Interval and Chain Homotopies

Use the standard topological simplices $|\Delta^n|$ of Lecture 7, Construction 7.1.11. The affine face inclusion d^i inserts a zero in the ith coordinate, and s^i adds adjacent coordinates. In particular, $|\Delta^1| = I$.

For a set S, write $\mathbb{Z}\{S\}$ for the free abelian group on S. Use the complex $\mathbb{Z}\{\Delta^1\}^{\mathrm{nd}}_\bullet$ from Lecture 7, Definition 7.1.15, whose degree-n group is free abelian on the set of nondegenerate n-simplices of Δ^1. Thus it has generators e_0, e_1 in degree zero and e_{01} in degree one, with $\partial e_{01} = e_1 - e_0$, and is zero in other degrees.

The tensor differential explains the sign in the degree-one component of its cylinder homotopy. For chain maps $f_0, f_1 : A_\bullet \to B_\bullet$, a chain homotopy from f_0 to f_1 is equivalently a chain map

$$h : A_\bullet \otimes \mathbb{Z}\{\Delta^1\}^{\mathrm{nd}}_\bullet \longrightarrow B_\bullet$$

whose restrictions to the endpoints are f_0, f_1. Indeed, for $\lambda \in A_n$, a degree-one map $s : A_\bullet \to B_{\bullet+1}$ determines the graded map h by

$$h(\lambda \otimes e_0) = f_0(\lambda), \qquad h(\lambda \otimes e_1) = f_1(\lambda), \qquad h(\lambda \otimes e_{01}) = (-1)^n s(\lambda).$$

With the tensor differential of Definition A.1.1, the chain-map identity on $\lambda \otimes e_{01}$ is

$$(-1)^n \partial_B s(\lambda) = (-1)^{n-1} s(\partial_A \lambda) + (-1)^n (f_1(\lambda) - f_0(\lambda)),$$

or $\partial_B s + s\partial_A = f_1 - f_0$. The identities on the two vertices say that f_0, f_1 are chain maps.

Remark A.1.3. For an additive category $\mathcal{A}$, write $\underline{\mathrm{Mor}}_{\mathrm{Ch}(\mathcal{A})}$ for the mapping complex, defined in the same way using $\mathrm{Mor}_{\mathcal{A}}$, with the products taken in abelian groups. Its zeroth homology gives the morphisms in the homotopy category:

$$\mathrm{Mor}_{K(\mathcal{A})}(A_\bullet, B_\bullet) = \mathrm{H}_0 \underline{\mathrm{Mor}}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet).$$

Indeed, cycles are chain maps and boundaries are differences of chain-homotopic maps.

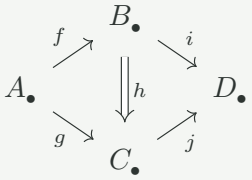
A.2 Homotopy Pushouts of Chain Complexes

A.2.1 The Comparison Map and Its Signs

A square with a chosen chain homotopy induces a comparison from the cone construction of Lecture 7. The degree-one component of the homotopy supplies the third term of that comparison.

The following square is the algebraic setting for the comparison. The endpoint maps and the chosen homotopy are retained as part of its data.

Definition A.2.1. A homotopy-commutative square of complexes consists of chain maps



and a specified chain map

$$h : A_\bullet \otimes \mathbb{Z}\{\Delta^1\}^{\mathrm{nd}}_\bullet \longrightarrow D_\bullet, \qquad h(\nu \otimes e_0) = if(\nu), \quad h(\nu \otimes e_1) = jg(\nu).$$

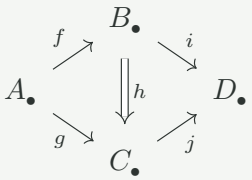
For $\nu \in A_n$, define its degree-one component by $s(\nu) = (-1)^n h(\nu \otimes e_{01})$. The chain-map identity for h is equivalent to

$$\partial_D s + s\partial_A = jg - if. \tag{18}$$

Thus h is a homotopy from if to jg, represented by the degree-one map s.

For the span $B_\bullet \xleftarrow{f} A_\bullet \xrightarrow{g} C_\bullet$, use the cone construction of Lecture 7, Definition 7.1.15, with $A_\bullet, B_\bullet, C_\bullet$ in place of the three singular chain complexes. Denote the resulting complex by $Z_\bullet$ and its structure maps by i_Z, j_Z, h_Z. Write s_Z for the degree-one component of h_Z. A degree-n element of $Z_\bullet$ is written (λ, μ, ν), where $\lambda \in B_n$, $\mu \in C_n$, and $\nu \in A_{n-1}$. The construction applies to arbitrary chain maps because its differential and homotopy identities use only the chain-map identities for f and g.

Proposition A.2.2. For a homotopy-commutative square



as in Definition A.2.1, there is a chain map

$$\Phi_h : Z_\bullet \to D_\bullet, \qquad \Phi_h(\lambda, \mu, \nu) = -i(\lambda) + j(\mu) + s(\nu). \tag{19}$$

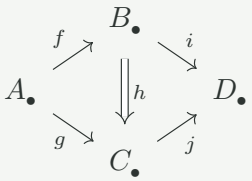
Its composites with the two structure maps i_Z, j_Z are i and j.

Proof. The chain-map condition is the homotopy identity (18). For $(\lambda, \mu, \nu) \in Z_n$, expand the two sides:

$$\begin{aligned}
\Phi_h \partial_Z(\lambda, \mu, \nu) &= -i\partial_B\lambda + j\partial_C\mu + (jg - if)(\nu) - s\partial_A\nu \\
&= -i\partial_B\lambda + j\partial_C\mu + \partial_D s(\nu) && \text{by (18)} \\
&= \partial_D(-i(\lambda) + j(\mu) + s(\nu)) && \text{since } i, j \text{ are chain maps} \\
&= \partial_D \Phi_h(\lambda, \mu, \nu).
\end{aligned}$$

Finally, $\Phi_h(-\lambda, 0, 0) = i(\lambda)$ and $\Phi_h(0, \mu, 0) = j(\mu)$, giving the required composites. □

Definition A.2.3. A homotopy-commutative square of complexes



is a *homotopy pushout square* if its comparison map Φ_h in Proposition A.2.2 is a quasi-isomorphism.

This is a condition in the homotopy theory of complexes with quasi-isomorphisms as equivalences. With the structure maps i_Z, j_Z, h_Z, the canonical complex $Z_\bullet$ is a homotopy pushout: its comparison map to itself is the identity. The square with rightmost complex $D_\bullet$ is a homotopy pushout when Φ_h is a quasi-isomorphism. The sign in its first structure map makes the comparison map restrict to the given maps i and j.

A.2.2 The Long Exact Homology Sequence

The shifted common complex occurs as a quotient of the double mapping cylinder. This quotient produces the connecting map, including its sign.

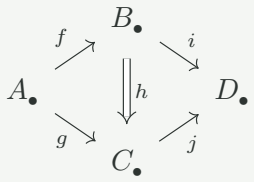
Apply the mapping-cone long exact sequence of Lecture 2 to $(f,g) : A_\bullet \to B_\bullet \oplus C_\bullet$. Let $k : B_\bullet \oplus C_\bullet \to Z_\bullet$ be the summand inclusion and let $p : Z_\bullet \to A_\bullet[1]$ be the projection onto the shifted summand.

For a cycle $\nu \in A_{n-1}$, the lift $(0,0,\nu) \in Z_n$ has boundary $(f(\nu), g(\nu), 0)$. Hence the boundary homomorphism out of $\mathrm{H}_n(A[1])$ is (f_*, g_*). The projection sends $[(\lambda,\mu,\nu)]$ to $[\nu]$ under the shift identification. We take the negative of this projection on homology, so that the convention agrees with the Mayer–Vietoris sequence in Lecture 7:

$$\delta_Z[(\lambda,\mu,\nu)] = -[\nu]. \tag{20}$$

Negating a homomorphism leaves its kernel and image unchanged, so this convention preserves exactness. The resulting class is independent of the chosen cycle representative because p is a chain map.

Theorem A.2.4. For a homotopy pushout square of complexes



there is a long exact sequence whose five consecutive terms are

$$\mathrm{H}_n(A) \xrightarrow{(f_*,g_*)} \mathrm{H}_n(B) \oplus \mathrm{H}_n(C) \xrightarrow{j_*-i_*} \mathrm{H}_n(D) \xrightarrow{\delta_D} \mathrm{H}_{n-1}(A) \xrightarrow{(f_*,g_*)} \mathrm{H}_{n-1}(B) \oplus \mathrm{H}_{n-1}(C). \tag{21}$$

Its connecting map is $\delta_D = \delta_Z(\Phi_{h,*})^{-1}$.

Proof. By Definition A.2.3, the homomorphism $\Phi_{h,*} : \mathrm{H}_n(Z) \to \mathrm{H}_n(D)$ is an isomorphism in every degree. Transport the mapping-cone exact sequence along these isomorphisms, using the convention (20). Formula (19) gives

$$\Phi_h k(\lambda,\mu) = j(\mu) - i(\lambda),$$

so the map into $\mathrm{H}_n(D)$ is $j_* - i_*$. The transported connecting map is the one stated, proving (21). □

A cycle $\zeta \in D_n$ can therefore be handled as follows. Choose a cycle $(\lambda,\mu,\nu) \in Z_n$ whose image under Φ_h is homologous to ζ. The cycle conditions and the image are

$$\partial_A \nu = 0, \qquad \partial_B \lambda = -f(\nu), \qquad \partial_C \mu = -g(\nu),$$
$$\zeta \sim -i(\lambda) + j(\mu) + s(\nu).$$

Then $\delta_D[\zeta] = -[\nu]$. The common chain ν records the boundary that has to be matched between the two pieces. The preceding projection argument proves that this class is independent of the representative in Z_n.

For the mapping-cone specialization $C_\bullet = 0$, the map from $\mathrm{H}_n(B)$ is induced by the negative of the usual summand inclusion. This is the same sign convention as in Lecture 7; the mapping-cone and suspension calculations themselves are given there and in Lecture 2.

References

[Stacks] The Stacks Project Authors, *The Stacks Project*, 2026. https://stacks.math.columbia.edu.

[May99] J. P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press, 1999. Further background for Lectures 5 and 7: Chapters 6, 10, and 13–15; Chapter 10, §7, p. 80 treats the double mapping cylinder.