Fibrations and the Homotopy Exact Sequence

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10.1 Lifting Homotopies

A map $p : Y \to X$ describes a space parametrized over X: the space lying over $x \in X$ is the fiber $Y_x = p^{-1}(x)$. A fibration is a parametrized space with a homotopy lifting property. This property governs how the fibers vary along paths in X, by allowing a deformation in the base to be followed upstairs from a prescribed initial lift. The lifts need not be unique, so this does not specify a canonical transport map between the fibers.

For a chosen point $y_0 \in Y$ over $x_0 \in X$, put $Z = Y_{x_0}$. The resulting fiber sequence has the order $Z \to Y \to X$, formally dual to the order $X \to Y \to Z$ in a cofiber sequence. Its homotopy exact sequence relates the fiber Z, the total space Y, and the base X.

We use the based homotopy groups of Lecture 9. Write $I = [0, 1]$, and give path spaces the compact-open topology. Since I is compact Hausdorff, a map from any parameter space into the compact-open path space is continuous exactly when its evaluation map is continuous. We use this exponential law when constructing families of paths.

10.1.1 The Homotopy Lifting Property

The lifting condition specifies both the deformation in the base and its initial value upstairs.

Definition 10.1.1. A map $p : Y \to X$ has the *homotopy lifting property* for a space K if every commutative square of solid arrows admits a dotted lift:

$$\begin{array}{ccc} K \times \{0\} & \xrightarrow{u} & Y \\ \downarrow & \overset{\widetilde{H}}{\nearrow} & \downarrow p \\ K \times I & \xrightarrow[H]{} & X. \end{array}$$

A *Hurewicz fibration* has this property for every space K. A *Serre fibration* has it for every disk D^m, including $m = 0$. Thus every Hurewicz fibration is a Serre fibration. In this lecture, the unqualified word fibration will mean a Serre fibration.

The condition for $m = 0$ is existence of path lifts with prescribed starting point. It does not assert uniqueness. For example, a path in X has many lifts under the product projection $X \times Z \to X$ when the fiber Z has nonconstant paths.

Lemma 10.1.2. Let $p : Y \to X$ be a Serre fibration and (K, A) a finite CW pair. Given a homotopy $H : K \times I \to X$ and a lift on

$$(K \times \{0\}) \cup (A \times I),$$

there is a lift on $K \times I$ extending the given lift.

Proof. Attach the cells of K relative to A, one at a time. For an m-cell, the required extension problem has prescribed lift on the bottom and sides of $\mathrm{D}^m \times I$. This subset is a closed m-disk in the boundary of the $(m+1)$-disk $\mathrm{D}^m \times I$. The pair consisting of the prism and its bottom and sides is homeomorphic to the pair consisting of the same prism and its bottom. For $m > 0$, rescale the cylinder to $\mathrm{D}^m \times [-1, 1]$ and radially identify its boundary with S^m. The bottom

and the bottom together with the sides become latitude caps with boundary heights $-1/\sqrt{2}$ and $1/\sqrt{2}$. An increasing piecewise linear change of the latitude coordinate taking the first height to the second, with the longitudinal direction unchanged, gives the required boundary homeomorphism. Extend it radially over the disk. For $m = 0$ no change is needed. Applying the disk lifting property after this change of coordinates gives the desired extension. The attaching pushout then combines it with the lift already constructed. Finitely many such extensions finish the proof. □

This relative version is important: based homotopies must keep the basepoint fixed, and comparisons of lifts must preserve prescribed faces of a cube.

10.1.2 Bundles and Examples

Local product coordinates provide a concrete way to construct lifts, even when no global product description exists.

Definition 10.1.3. A *fiber bundle* with fiber Z is a map $p : Y \to X$ such that every point of X has an open neighborhood U and a homeomorphism

$$\phi : p^{-1}(U) \longrightarrow U \times Z, \qquad \mathrm{pr}_1 \, \phi = p.$$

Such a homeomorphism is a local trivialization. The fibers $p^{-1}(x)$ are therefore homeomorphic to Z.

Theorem 10.1.4. Every fiber bundle is a Serre fibration.

Proof. Consider $H : I^m \times I \to X$ and a lift at time zero. The inverse images under H of trivializing neighborhoods cover the compact cube. Choose a finite cubical subdivision, with a common sufficiently small mesh in all coordinates, so that every closed small cube is carried into one trivializing neighborhood. This is possible by the Lebesgue number lemma.

Proceed through the time intervals of this subdivision in order. On one time interval, proceed through the spatial cells in increasing dimension. Suppose the lift has been constructed on the bottom and sides of a spatial cell times this interval. Choose a trivialization containing its entire image under H. In this chart the prescribed lift is a map into Z on the bottom and sides. A prism retracts onto its bottom and sides. For the unit prism $\mathrm{D}^d \times I$, an explicit retraction is $r(x, t) = (\lambda x, 2 + \lambda(t - 2))$, where $\lambda = 2/\max\{2\|x\|, 2 - t\}$. It fixes the bottom and sides and lands in them; rescale the formula for the small prism. For a cubical cell use the maximum norm. Extend the fiber-coordinate map by composing with this retraction. Pair it with H and apply the inverse trivialization. This extends the lift and agrees with every previously constructed face. The finite pasting lemma supplies a continuous lift on each slab and then on the whole cube. Cubes and disks are homeomorphic, so this proves the Serre condition. □

Example 10.1.5. The exponential map $e : \mathbb{R} \to \mathrm{S}^1$, defined by $e(t) = \exp(2\pi i t)$, is a fiber bundle with discrete fiber $\mathbb{Z}$. On an open arc shorter than a full circle, choose a continuous argument. Its translates by integers give a trivialization. Lecture 11 will use the uniqueness of these lifts to compute $\pi_1(\mathrm{S}^1, 1)$.

Lemma 10.1.6. If $p : Y \to X$ is a Serre fibration and $f : T \to X$ is continuous, then the pullback projection $T \times_X Y \to T$ is a Serre fibration.

Proof. Compose a lifting problem over T with f and lift it to Y. The resulting lift and the original map to T have the same composite to X. The universal property of the pullback gives the required lift to $T \times_X Y$. □

10.2 The Homotopy Exact Sequence

Fix $y_0 \in Y$, put $x_0 = p(y_0)$, and let $Z = p^{-1}(x_0)$ with basepoint y_0. Write $i : Z \to Y$ for inclusion. The groups in the sequence depend on these choices, even when we suppress repeated basepoints in explanatory prose.

10.2.1 The Connecting Map

The connecting map records the failure of a lifted sphere to close up upstairs: the remaining face lies in the fiber.

Construction 10.2.1. For $n \geq 1$, represent a class in $\pi_n(X, x_0)$ by a map

$$u : (I^{n-1} \times I, \partial(I^{n-1} \times I)) \longrightarrow (X, x_0).$$

Lift u starting with the constant lift y_0 on the bottom and keeping the lift equal to y_0 on the sides, using Lemma 10.1.2. The top face of the lift is a based map $I^{n-1} \to Z$. Its class is

$$\partial : \pi_n(X, x_0) \longrightarrow \pi_{n-1}(Z, y_0).$$

For $n = 1$, this means the path component of the endpoint of a lifted loop.

Lemma 10.2.2. The connecting map is independent of the lift and of the representative. It is a homomorphism for $n \geq 2$ and is natural for maps of based fibrations.

Proof. Let u_s be a based homotopy between representatives, and choose lifts at both parameter endpoints. Apply relative lifting in the last cube coordinate t, with parameter pair

$$K = I^{n-1} \times I_s, \qquad A = (\partial I^{n-1} \times I_s) \cup (I^{n-1} \times \{0, 1\}).$$

Prescribe the constant lift on $K \times \{0\}$ and on the x-side faces, and the two chosen lifts on the s-end faces. These data agree on intersections. Lemma 10.1.2 extends them, and restriction to $t = 1$ gives a based homotopy in Z between the two top maps. Taking u_s constant also proves independence of the chosen lift.

For $n \geq 2$, concatenate representatives in one of the first $n-1$ coordinates. Their chosen lifts concatenate in that coordinate as well, and their top faces represent the product in $\pi_{n-1}(Z, y_0)$. Thus ∂ is a homomorphism. Finally, composing a chosen lift with a map of total spaces gives a lift of the corresponding map downstairs. Independence of choices proves naturality. □

10.2.2 Exactness and Components

A class vanishes after one map precisely when the preceding map can supply it. The cubical description makes each of these assertions a lifting or filling problem.

Theorem 10.2.3. For a based Serre fibration, the following sequence is exact:

$$\pi_n(Z, y_0) \xrightarrow{i_*} \pi_n(Y, y_0) \xrightarrow{p_*} \pi_n(X, x_0) \xrightarrow{\partial} \pi_{n-1}(Z, y_0) \xrightarrow{i_*} \pi_{n-1}(Y, y_0) \qquad (n \geq 2).$$

It continues as a sequence of pointed sets

$$\pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(X, x_0) \xrightarrow{\partial} \pi_0(Z) \xrightarrow{i_*} \pi_0(Y) \xrightarrow{p_*} \pi_0(X).$$

Exactness at a pointed set means that the image of the preceding map is the inverse image of the distinguished point under the next map.

Proof. At $\pi_n(Y, y_0)$, a class coming from Z projects to the constant class. Conversely, if a based map $v : I^n \to Y$ projects to a nullhomotopic map, lift a based nullhomotopy of pv starting from v and fixed on the boundary. Its final map lies in Z and is based homotopic in Y to v.

At $\pi_n(X, x_0)$, a class lifted by a based map into Y has constant endpoint map in Z, so its boundary is zero. Conversely, if the top face of a lift of u is nullhomotopic in Z, append that nullhomotopy to the lift in the last coordinate. The result is a based cube in Y. Its projection is u followed by a constant slab, hence represents the original class.

At $\pi_{n-1}(Z, y_0)$, the top face of a lifted cube bounds that cube in Y, with all other faces at y_0. Thus its image in Y is zero. Conversely, a based nullhomotopy in Y of a map into Z, read from the constant map to that map, projects to a cube in X whose entire boundary is x_0. The original nullhomotopy is a lift, and its top face is the given map into Z.

The same arguments in dimension one use paths rather than higher cubes. In particular, a lifted loop ends in the component of y_0 in Z exactly when one can append a path in Z to obtain a loop in Y. At $\pi_0(Z)$, a point of Z is connected to y_0 in Y exactly when it is the endpoint of a path whose projection is a loop at x_0. At $\pi_0(Y)$, a point whose image is connected to x_0 can be joined, by lifting such a path, to a point of Z. These prove the remaining assertions. □

The component terms are generally not groups. Exactness there does not justify subtracting components or applying an abelian-group argument. If Z is path connected, however, the sequence immediately shows that $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is surjective.

10.3 The Path–Loop Fibration

10.3.1 Paths Beginning at a Basepoint

The most useful universal example comes from allowing all paths that start at one fixed point.

Construction 10.3.1. For a based space (X, x_0), define

$$PX = \{\gamma : I \to X \mid \gamma(0) = x_0\}, \qquad \Omega X = \{\gamma \in PX \mid \gamma(1) = x_0\}.$$

The endpoint map $\mathrm{ev}_1 : PX \to X$ has fiber ΩX over x_0. Both path spaces have the constant path as basepoint.

Proposition 10.3.2. The endpoint map $\mathrm{ev}_1 : PX \to X$ is a Hurewicz fibration, and PX is contractible through a contraction fixing the constant path.

Proof. Suppose $H : K \times I \to X$ is a homotopy and $u : K \to PX$ satisfies $u(k)(1) = H(k, 0)$. Define a path $\widetilde{H}(k, t)$ by first following $u(k)$ and then following $H(k, -)$ up to time t. To make its value at $t = 0$ exactly $u(k)$, use the formula

$$\widetilde{H}(k,t)(s) = \begin{cases} u(k)((1+t)s), & 0 \le s \le (1+t)^{-1}, \\ H(k, (1+t)s - 1), & (1+t)^{-1} \le s \le 1. \end{cases}$$

The two formulas agree at their common endpoint, and their values at $s = 1$ equal $H(k, t)$. Continuity follows from the pasting lemma for the evaluation map and the compact-open exponential law, since I is compact Hausdorff. This gives the required lift for every K.

The formula $C(\gamma, t)(s) = \gamma((1 - t)s)$ contracts PX to the constant path and fixes that path throughout. □

10.3.2 Looping Lowers the Homotopy Degree

A contractible total space leaves only the relation between the base and the fiber in the exact sequence.

Corollary 10.3.3. For $n \geq 2$, the path–loop fibration gives a natural isomorphism

$$\partial : \pi_n(X, x_0) \xrightarrow{\cong} \pi_{n-1}(\Omega X, \mathrm{const}_{x_0}).$$

For $n = 1$, it gives the natural bijection between $\pi_1(X, x_0)$ and the set $\pi_0(\Omega X)$.

Proof. Every positive homotopy group of PX vanishes because of its based contraction. The exact sequence therefore makes ∂ injective and surjective in positive degrees. At the component end, PX is path connected, and the same lifting argument identifies components of ΩX exactly with based homotopy classes of loops in X. □

A map $I^{n-1} \to \Omega X$ carrying ∂I^{n-1} to the constant loop is equivalently a map $I^{n-1} \times I \to X$ that is constant on the full boundary. This adjunction is the direct geometric explanation for the degree shift. In particular, looping does not simplify a space pointwise; it packages higher-dimensional based homotopies as lower-dimensional maps into a new space.

Corollary 10.3.4. For every $n \geq 2$, one has $\pi_n(\mathrm{S}^1, 1) = 0$.

Proof. Apply the fibration sequence to $\mathbb{Z} \to \mathbb{R} \xrightarrow{e} \mathrm{S}^1$. The based contraction of $\mathbb{R}$ makes its positive homotopy groups zero. Every positive homotopy group of the discrete space $\mathbb{Z}$ is zero, since the image of a connected cube is a point. Thus the exact segment

$$\pi_n(\mathbb{R}, 0) \longrightarrow \pi_n(\mathrm{S}^1, 1) \longrightarrow \pi_{n-1}(\mathbb{Z}, 0)$$

has zero outer terms for $n \geq 2$. □

10.4 The Hopf Fibration

10.4.1 The Bundle over the Projective Line

Regard S^3 as the unit sphere in $\mathbb{C}^2$. The complex projective line $\mathbb{CP}^1$ is the space of complex lines in $\mathbb{C}^2$, with the quotient topology. The quotient map below sends a unit vector to its line:

$$h : \mathrm{S}^3 \longrightarrow \mathbb{CP}^1, \qquad (z_0, z_1) \longmapsto [z_0 : z_1].$$

Its fibers are precisely the orbits of the scalar action of S^1 on S^3.

Proposition 10.4.1. The map h is a fiber bundle with fiber S^1, and $\mathbb{CP}^1$ is homeomorphic to S^2.

Proof. On the open chart $U_0 = \{[z_0 : z_1] \mid z_0 \neq 0\}$, use the coordinate $w = z_1/z_0$. A section is

$$s_0(w) = \frac{(1, w)}{\sqrt{1 + |w|^2}}.$$

The map $U_0 \times \mathrm{S}^1 \to h^{-1}(U_0)$ given by $(w, \lambda) \mapsto \lambda s_0(w)$ is a homeomorphism over U_0. Its inverse takes a unit vector to $(z_1/z_0, z_0/|z_0|)$. On $U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\}$, the analogous section is $(v, 1)/\sqrt{1 + |v|^2}$, where $v = z_0/z_1$. These two trivializations prove the bundle assertion.

The map

$$(z_0, z_1) \longmapsto (2\,\mathrm{Re}(\overline{z}_0 z_1),\ 2\,\mathrm{Im}(\overline{z}_0 z_1),\ |z_0|^2 - |z_1|^2)$$

takes values in the unit sphere in $\mathbb{R}^3$ and is constant on scalar orbits. It therefore induces a continuous map $\mathbb{CP}^1 \to \mathrm{S}^2$. For a point $(x, y, z) \in \mathrm{S}^2$ with $z \neq -1$, the unique preimage has chart coordinate $w = (x + iy)/(1 + z)$. The remaining point $(0, 0, -1)$ has preimage $[0 : 1]$. Thus the induced map is bijective. The domain is compact as a quotient of S^3, and the target is Hausdorff. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. □

With this identification, $h : \mathrm{S}^3 \to \mathrm{S}^2$ is the *Hopf map*. Choose $y_0 = (1, 0)$ and its image x_0 as basepoints, and identify the fiber with S^1 by $\lambda \mapsto (\lambda, 0)$.

10.4.2 Computing the Third Homotopy Group

The circle fiber has no higher homotopy groups, so the Hopf bundle transfers the third homotopy group of its total space to its base.

Theorem 10.4.2. The Hopf map induces an isomorphism

$$h_* : \pi_3(\mathrm{S}^3, y_0) \xrightarrow{\cong} \pi_3(\mathrm{S}^2, x_0).$$

Consequently $\pi_3(\mathrm{S}^2, x_0) \cong \mathbb{Z}$, with generator represented by h after an orientation of S^3 is chosen.

Proof. The relevant five terms of the Hopf fibration sequence are

$$\pi_3(\mathrm{S}^1, 1) \longrightarrow \pi_3(\mathrm{S}^3, y_0) \xrightarrow{h_*} \pi_3(\mathrm{S}^2, x_0) \longrightarrow \pi_2(\mathrm{S}^1, 1) \longrightarrow \pi_2(\mathrm{S}^3, y_0).$$

The first and fourth terms vanish by Corollary 10.3.4. Exactness proves that h_* is an isomorphism.

Lemma 9.5.1 proves that S^3 is 2-connected. Corollary 9.5.2 applies the stated Hurewicz theorem to give

$$\pi_3(\mathrm{S}^3, y_0) \xrightarrow{\cong} \mathrm{H}_3(\mathrm{S}^3; \mathbb{Z}) = \mathbb{Z}.$$

The identity map represents the oriented generator under this isomorphism. Its image under h_* is the homotopy class of h. This last identification uses the Hurewicz theorem whose proof, as specified in the syllabus, is deferred to Math 616; the fibration argument above is proved in full here. □

This is a first indication that homotopy groups differ sharply from homology groups. Although $\mathrm{H}_3(\mathrm{S}^2; \mathbb{Z}) = 0$, the group $\pi_3(\mathrm{S}^2, x_0)$ is infinite cyclic. A map from a higher-dimensional sphere can carry essential homotopy information that its map on homology does not detect.

For further reading, see May, *A Concise Course in Algebraic Topology*, Chapter 7 for the lifting viewpoint and Chapter 9, Sections 3–4, pp. 66–67, for the fibration sequence and Hopf example. May works with Hurewicz fibrations; the disk-lifting arguments above isolate the weaker Serre hypothesis needed for homotopy groups. A supplementary comparison is Hatcher, *Algebraic Topology*, Theorem 4.41, pp. 376–377, and Proposition 4.48, pp. 379–380, for the Serre sequence and the local bundle argument.