# Homotopy Groups and the Fundamental Group Action

Algebraic Topology I | Fall 2026 | Companion to Lecture 9

Use the definitions and results of Lecture 9. Concatenation $\alpha * \beta$ traverses $\alpha$ first. A path $\alpha$ from $x$ to $y$ induces transport $\tau_\alpha : \pi_n(X, x) \to \pi_n(X, y)$. There are four exercises.

## Exercise 1. Changing the Transport Path

Let $\alpha, \beta : I \to X$ run from $x$ to $y$, and let $n \geq 1$. Write $\bar{\beta}(t) = \beta(1 - t)$.

(a) Prove that $\tau_\beta^{-1} \tau_\alpha = \tau_{\alpha * \bar{\beta}}$. Deduce that transport from $x$ to $y$ is independent of the path if every loop at $x$ acts trivially on $\pi_n(X, x)$.

(b) For $n = 1$, prove that path independence holds if and only if $\pi_1(X, x)$ is abelian. Use the formula $\tau_\gamma([f]) = [\bar{\gamma} * f * \gamma]$, rather than changing the convention for concatenation.

(c) Let $q : X \to Y$ be a homotopy equivalence. Choose $x \in X$ and a homotopy inverse $r$. Explain how the trace of a homotopy $rq \simeq \mathrm{id}_X$ supplies the basepoint correction in the proof that $q_* : \pi_n(X, x) \to \pi_n(Y, q(x))$ is an isomorphism. Draw the naturality square for transport that is used in this correction.

## Exercise 2. Homotopy Groups of a Topological Group

Let $G$ be a path-connected topological group with identity $e$. Represent an element of $\pi_n(G, e)$ by a map $f : I^n \to G$ sending $\partial I^n$ to $e$.

(a) Pointwise multiplication defines a second operation on $\pi_n(G, e)$. Verify the interchange law with concatenation in the first coordinate. Deduce that $\pi_1(G, e)$ is abelian.

(b) Let $\alpha : I \to G$ be a loop at $e$. Apply the moving-boundary lemma of Lecture 9 to $H(u, t) = \alpha(t) f(u)$. Prove that $\tau_\alpha$ is the identity on $\pi_n(G, e)$ for every $n \geq 1$.

(c) Prove that the map from based homotopy classes of maps $S^n \to G$ to free homotopy classes is a bijection. State where path connectedness is used.

## Exercise 3. The First Nonzero Homotopy Group of a Product of Spheres

Let $2 \leq p \leq q$ and give $S^p \times S^q$ its product basepoint.

(a) Prove directly from cube representatives that the two projections induce an isomorphism

$$\pi_k(S^p \times S^q) \longrightarrow \pi_k(S^p) \times \pi_k(S^q)$$

for every $k \geq 1$.

(b) Compute these groups for $1 \leq k \leq p$. Treat $p = q$ separately from $p < q$.

(c) Compute $\mathrm{H}_p(S^p \times S^q; \mathbb{Z})$ from its product cell structure. Use naturality of the Hurewicz map with respect to the inclusions and projections to identify the Hurewicz isomorphism in degree $p$, including its generators.

## Exercise 4. Degree and Homotopy Equivalences

Fix $n \geq 2$ and orientations on $S^n$. You may use the connectivity of spheres, the Hurewicz theorem, and the description of free homotopy classes by basepoint transport from Lecture 9.

(a) Prove that two based self-maps are based homotopic if and only if they have the same degree. Explain why the same classification holds for free homotopy classes.

(b) Prove that a self-map $f : S^n \to S^n$ is a homotopy equivalence if and only if $\deg(f) = 1$ or $-1$. In the reverse implication, construct a homotopy inverse using degree classification, rather than assuming that a homology isomorphism is a homotopy equivalence.

(c) Let $a : S^n \to S^n$ be the antipodal map. Determine for which $n$ it is homotopic to the identity. Explain why the answer is independent of whether homotopies are required to preserve a chosen basepoint, after replacing $a$ by a based representative of its free homotopy class.

# Fibrations and Homotopy Exact Sequences

Algebraic Topology I | Fall 2026 | Companion to Lecture 10

Use the homotopy exact sequence of a fibration from Lecture 10. All spaces and maps are based unless stated otherwise. There are four exercises.

## Exercise 1. A Fibration with Contractible Total Space

Let $p : E \to B$ be a fibration, with fiber $F = p^{-1}(b)$ and chosen $e \in F$. Assume that $E$ is contractible and $B$ is path connected.

(a) For $k \geq 2$, prove that the connecting map $\pi_k(B, b) \to \pi_{k-1}(F, e)$ is an isomorphism.

(b) Show that $\pi_1(B, b) \to \pi_0(F)$ is a bijection of pointed sets. Do not call it a group isomorphism without first specifying a group structure on $\pi_0(F)$.

(c) Apply this to evaluation $P_bB \to B$ on paths beginning at $b$. Give an explicit contraction of $P_bB$, and identify its fiber. Describe the connecting map in terms of lifting a representative cube.

## Exercise 2. The Hopf Map and Maps into the Two-Sphere

Let $h : S^3 \to S^2$ be the Hopf fibration with fiber $S^1$. You may use $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_k(S^1) = 0$ for $k \geq 2$.

(a) Write the exact sequence that proves $h_* : \pi_3(S^3) \to \pi_3(S^2)$ is an isomorphism. Identify a generator of $\pi_3(S^2)$ after choosing an orientation on $S^3$.

(b) If $f : S^3 \to S^3$ has degree $d$, determine the class of $h \circ f$ in $\pi_3(S^2)$. Prove that every based map $S^3 \to S^2$ is based homotopic to such a composite.

(c) Show that every map $S^3 \to S^2$ induces zero on reduced integral homology, but that infinitely many of these maps are not nullhomotopic. Explain why this does not contradict the Hurewicz theorem.

## Exercise 3. Higher Complex Projective Spaces

For $n \geq 1$, let $p : S^{2n+1} \to \mathbb{CP}^n$ send a unit vector in $\mathbb{C}^{n+1}$ to its complex line.

(a) On the chart where the $j$th homogeneous coordinate is nonzero, choose the unit representative whose $j$th coordinate is positive real. Use it to construct a local trivialization of $p$ with fiber $S^1$.

(b) Use the homotopy exact sequence to prove that $\mathbb{CP}^n$ is simply connected and compute $\pi_2(\mathbb{CP}^n)$.

(c) For $3 \leq k \leq 2n$, compute $\pi_k(\mathbb{CP}^n)$. Compute $\pi_{2n+1}(\mathbb{CP}^n)$ as well. The interval in the first sentence is empty when $n = 1$.

(d) Explain why the inclusion $\mathbb{CP}^1 \to \mathbb{CP}^n$ induces an isomorphism on $\pi_2$. Use the natural map between the two circle bundles and naturality of their connecting homomorphisms.

## Exercise 4. Products and a Section of a Fibration

Let $F \to E \xrightarrow{p} B$ be a fibration of path-connected based spaces. Suppose that $p$ admits a based section $s : B \to E$.

(a) Prove that every connecting map $\pi_k(B) \to \pi_{k-1}(F)$ is zero for $k \geq 2$.

(b) For $k \geq 2$, prove that the fiber inclusion and the section induce an isomorphism

$$\pi_k(F) \oplus \pi_k(B) \longrightarrow \pi_k(E).$$

(c) In degree one, prove instead that $\pi_1(E)$ is a semidirect product of $\pi_1(F)$ and $\pi_1(B)$. Define the action using conjugation by $s_*\pi_1(B)$, and explain why a direct product need not result.

(d) Take $F = S^1$ and let $E$ be the mapping torus of reflection of $S^1$, using a fixed point to define the section over $B = S^1$. Compute the action in part (c), and give a presentation for $\pi_1(E)$.

# Fundamental Groups and Attaching Maps

Algebraic Topology I | Fall 2026 | Companion to Lecture 11

Use Lecture 11 and the homology computations of Lecture 8. All displayed fundamental groups use the indicated vertex or a chosen point in the connected space. There are four exercises.

## Exercise 1. A Disk Attached to a Bouquet

Let $X = S^1 \vee S^1$ with oriented loops $a, b$. For integers $m, n \geq 1$, attach a disk along the loop $a^m b^n$ and call the resulting cell complex $Y_{m,n}$.

(a) Use the disk-attachment form of van Kampen to give a presentation of $\pi_1(Y_{m,n})$. Describe the normal subgroup that is divided out; do not replace normal closure by the cyclic subgroup generated by the attaching word.

(b) Compute the abelianization of this presentation and compare it with a direct cellular calculation of $\mathrm{H}_1(Y_{m,n}; \mathbb{Z})$ and $\mathrm{H}_2(Y_{m,n}; \mathbb{Z})$.

(c) Show that $Y_{1,n}$ has infinite cyclic fundamental group. For $Y_{2,2}$, construct a surjection from its fundamental group onto the nonabelian group generated by two reflections of the real line. Conclude that the fundamental group of $Y_{2,2}$ is not determined by its abelianization alone.

## Exercise 2. The Klein Bottle Group

Let $K$ be the cell complex with one vertex, edges $a, b$, and one disk attached along $aba^{-1}b$.

(a) Give a presentation of $\pi_1(K)$. Show that every element has a representative $b^r a^s$ with $r, s \in \mathbb{Z}$.

(b) On $\mathbb{R}^2$, define $A(x, y) = (x + 1, -y)$ and $B(x, y) = (x, y + 1)$. Verify the relation $ABA^{-1} = B^{-1}$. Use this action to prove uniqueness of the normal form in part (a).

(c) Compute the center of $\pi_1(K)$ and its abelianization. Compare the latter with $\mathrm{H}_1(K; \mathbb{Z})$.

## Exercise 3. Punctured Planes and Attaching Higher Cells

Choose distinct points $z_1, \dots, z_r$ in $\mathbb{R}^2$, where $r \geq 1$.

(a) Construct a graph deformation retract of $\mathbb{R}^2 \setminus \{z_1, \dots, z_r\}$, or give a van Kampen decomposition that computes its fundamental group. Specify the paths from a basepoint to the small circles around the punctures used to choose free generators.

(b) A map $S^{d-1} \to X$ attaches a $d$-cell to a path-connected cell complex $X$. For $d \geq 3$, prove that the induced map on fundamental groups is an isomorphism. State why the analogous assertion fails for $d = 2$.

(c) Starting with a bouquet of $r$ circles, attach any collection of cells of dimensions at least $3$, with the usual cell-complex topology. Prove that its fundamental group remains free on $r$ generators. You may use that each compact subset of a cell complex is contained in a finite subcomplex. Explain how this reduces loops and homotopies to the finite case.

## Exercise 4. Winding Number and Polynomial Perturbations

Let $p(z) = z^d + c_{d-1}z^{d-1} + \cdots + c_0$ be a monic complex polynomial of degree $d \geq 1$. Choose $R > 0$ so that

$$\sum_{j=0}^{d-1} |c_j| R^j < R^d.$$

(a) Prove that the loops $t \mapsto p(Re^{2\pi it})$ and $t \mapsto R^d e^{2\pi idt}$ are homotopic in $\mathbb{C} \setminus \{0\}$ by the straight-line homotopy. Compute their winding number around zero.

(b) Deduce that $p$ has a zero in $\{z : |z| \leq R\}$. Use extension over the disk, rather than a theorem counting polynomial roots.

(c) More generally, let $f, g : \{z : |z| \leq R\} \to \mathbb{C}$ be continuous and satisfy $|f - g| < |f|$ on the boundary. Prove that their boundary loops have the same winding number. If this number is nonzero, prove that both $f$ and $g$ have a zero in the disk.

# Covering Spaces and Free Subgroups

Algebraic Topology I | Fall 2026 | Companion to Lecture 12

Use Lecture 12. A graph is regarded as a one-dimensional cell complex. In a directed labeled covering of a bouquet, traversing an edge backward reads the inverse of its label. There are four exercises.

## Exercise 1. Lifting a Circle-Valued Map

Let $X$ be path connected and locally path connected, let $x \in X$, and let $f : X \to S^1$ satisfy $f(x) = 1$. For $m \geq 1$, let $p_m : S^1 \to S^1$ be $p_m(z) = z^m$.

(a) Prove that there is a lift $g : X \to S^1$ with $g(x) = 1$ and $p_m g = f$ if and only if $f_*\pi_1(X, x) \subseteq m\mathbb{Z}$. Explain why the based lift is unique.

(b) Prove that $f$ has a lift to $\mathbb{R}$ under $t \mapsto e^{2\pi i t}$ if and only if $f_* = 0$. Deduce that this is also equivalent to $f$ being nullhomotopic.

(c) Take $X = S^1$ and let $f$ have degree $d$. Determine when it has an $m$th root as a circle-valued map, and count its distinct lifts when the value at the basepoint is not prescribed.

## Exercise 2. A Three-Sheeted Cover of a Bouquet

Let $X = S^1 \vee S^1$ have oriented edges labeled $a, b$. Construct a labeled graph $Y$ with vertices $0, 1, 2$. The outgoing $a$-edge at $i$ ends at $i + 1$ modulo $3$. The outgoing $b$-edges interchange $0$ and $1$ and fix $2$. Map every edge homeomorphically onto the edge of $X$ with the same label.

(a) Verify that the resulting map $p : Y \to X$ is a connected three-sheeted covering.

(b) Choose as a maximal tree the $a$-edges from $0$ to $1$ and from $1$ to $2$. For each of the four remaining oriented edges, form the loop that travels from $0$ along the tree to its initial vertex, traverses the edge, and returns along the tree. Read their labels to give a free basis of $p_*\pi_1(Y, 0) \subseteq \langle a, b \rangle$.

(c) Compute the deck transformation group by finding the permutations of $\{0, 1, 2\}$ that commute with both edge permutations. Determine whether this cover is regular. Compare with the index and normalizer of the subgroup in part (b).

## Exercise 3. An Infinite-Index Subgroup with Infinite Rank

Let $X = S^1 \vee S^1$ again, and let $\varphi : \langle a, b\rangle \to \mathbb{Z}$ be defined by $\varphi(a) = 1$ and $\varphi(b) = 0$. Form a labeled graph with vertices the integers, an $a$-edge from $j$ to $j+1$, and a $b$-loop at each $j$.

(a) Verify that this graph is a connected covering of $X$. Show that the subgroup determined by the vertex $0$ is $\ker\varphi$ by following the endpoint of a lifted word.

(b) Use the line of $a$-edges as a maximal tree. Prove that the words $a^j b a^{-j}$, for $j \in \mathbb{Z}$, freely generate $\ker\varphi$.

(c) Compute the deck group. Explain why the finite-index rank formula cannot be applied by replacing the finite index with an unspecified infinite cardinal and then performing integer subtraction.

## Exercise 4. Intermediate Circle Covers

For a fixed $n \geq 1$, consider $p_n : S^1 \to S^1$. An intermediate connected cover is a factorization into covering maps

$$S^1 \xrightarrow{q} Y \xrightarrow{r} S^1, \qquad rq = p_n,$$

with chosen compatible basepoints. Intermediate covers are identified by based isomorphisms commuting with this factorization.

(a) Classify these factorizations using the subgroups $n\mathbb{Z} \subseteq H \subseteq \mathbb{Z}$. For every divisor $d$ of $n$, identify $q$ and $r$ as power maps and compute their numbers of sheets.

(b) Compute the deck groups of all three maps and the homomorphism between deck groups associated with each intermediate cover. Describe its kernel.

(c) For $n = 12$, draw the subgroup-inclusion diagram and the corresponding diagram of intermediate covers. Mark each covering arrow with its number of sheets, and explain why a smaller subgroup corresponds to a cover with more sheets.