Algebraic Topology I: Homework 1

Additive and abelian categories, chain complexes, and chain homotopies

Throughout, an *abelian category* means an additive category with all kernels and cokernels in which every monomorphism is a kernel and every epimorphism is a cokernel. In an arbitrary abelian category, give arguments using morphisms and universal properties.

Use homological grading: a chain complex has differentials $d_n : C_n \to C_{n-1}$ satisfying $d_{n-1}d_n = 0$. A chain homotopy between chain maps $f, g : C \to D$ consists of morphisms $h_n : C_n \to D_{n+1}$ satisfying

$$f_n - g_n = d^D_{n+1}h_n + h_{n-1}d^C_n.$$

Exercise 1. Mono, epi, and iso

Let $\mathcal{C}$ be a category. A morphism $f : X \to Y$ is called:

- a *monomorphism* if, for every object T and morphisms $u, v : T \to X$, the equality $fu = fv$ implies $u = v$;
- an *epimorphism* if, for every object T and morphisms $u, v : Y \to T$, the equality $uf = vf$ implies $u = v$;
- an *isomorphism* if there exists a morphism $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$.

(a) Prove that the inverse of an isomorphism is unique, and that every isomorphism is both a monomorphism and an epimorphism. In the category of sets, prove that monomorphisms are precisely injections, epimorphisms are precisely surjections, and isomorphisms are precisely bijections.

(b) Let **CRing** be the category of commutative rings with identity and identity-preserving ring homomorphisms. Show that the inclusion

$$\mathbb{Z} \hookrightarrow \mathbb{Q}$$

is both a monomorphism and an epimorphism, but is not an isomorphism. Thus an epimorphism of rings need not be surjective.

(c) Let **Haus** be the category of Hausdorff topological spaces and continuous maps. Give $\mathbb{Q}$ and $\mathbb{R}$ their usual topologies. Show that the inclusion

$$\mathbb{Q} \hookrightarrow \mathbb{R}$$

is both a monomorphism and an epimorphism in **Haus**, but is not an isomorphism. Is this inclusion still an epimorphism in the category **Top** of all topological spaces and continuous maps? Justify your answer.

(d) Let $\mathcal{A}$ be an abelian category. Prove that a morphism in $\mathcal{A}$ is an isomorphism if and only if it is both a monomorphism and an epimorphism.

Exercise 2. Coimage and image

Let $f : X \to Y$ be a morphism in an abelian category. Define

$$\operatorname{coim}(f) = \operatorname{coker}(\ker f), \qquad \operatorname{im}(f) = \ker(\operatorname{coker} f).$$

Let $p : X \to \operatorname{coim}(f)$ and $i : \operatorname{im}(f) \to Y$ be the canonical morphisms.

(a) Prove that there is a unique morphism

$$\bar{f} : \operatorname{coim}(f) \longrightarrow \operatorname{im}(f), \qquad f = i \circ \bar{f} \circ p.$$

(b) Prove that $\bar{f}$ is an isomorphism.

Exercise 3. Matrix algebra in additive categories

Let $\mathcal{C}$ be an additive category. For a biproduct $X_1 \oplus X_2$, write

$$\iota_j^X : X_j \longrightarrow X_1 \oplus X_2, \qquad \rho_i^X : X_1 \oplus X_2 \longrightarrow X_i$$

for its canonical inclusion and projection maps. The projections are the product structure maps, and the inclusions are characterized by

$$\rho_i^X \iota_j^X = \begin{cases} 1_{X_i}, & i = j, \\ 0, & i \neq j. \end{cases}$$

The inclusions also make $X_1 \oplus X_2$ a coproduct. Use analogous notation for $Y_1 \oplus Y_2$.

(a) Show that a morphism

$$f : X_1 \oplus X_2 \longrightarrow Y_1 \oplus Y_2$$

is uniquely determined by its four components

$$f_{ij} = \rho_i^Y f \iota_j^X.$$

Express f in terms of these components, and prove that composition corresponds to matrix multiplication.

(b) Let $\mathcal{F}$ be the category of finitely generated free abelian groups, with all group homomorphisms as morphisms. Show that $\mathcal{F}$ is additive. Prove that

$$2 : \mathbb{Z} \longrightarrow \mathbb{Z}$$

is both a monomorphism and an epimorphism in $\mathcal{F}$, but is not an isomorphism.

(c) Compute the kernel and cokernel of this morphism *in* $\mathcal{F}$. Then compute its image, coimage, and the canonical morphism

$$\operatorname{coim}(2) \longrightarrow \operatorname{im}(2).$$

Explain why $\mathcal{F}$ is not abelian.

Exercise 4. Splittings and idempotents

Let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

be a short exact sequence in an abelian category $\mathcal{A}$.

(a) Suppose $s : C \to B$ satisfies $ps = 1_C$. Construct $r : B \to A$ such that

$$ri = 1_A, \qquad rs = 0, \qquad ir + sp = 1_B.$$

Deduce that

$$[\, i \;\; s \,] : A \oplus C \longrightarrow B$$

is an isomorphism, and describe its inverse.

(b) Prove that p has a section if and only if i has a retraction. Thus either condition characterizes a split short exact sequence.

(c) Let $e : X \to X$ satisfy $e^2 = e$. Construct an isomorphism

$$X \cong \operatorname{im}(e) \oplus \ker(e)$$

under which e has matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Exercise 5. Chain complexes form an abelian category

Let $\mathcal{A}$ be an abelian category, and write $\mathrm{Ch}(\mathcal{A})$ for its category of chain complexes and chain maps.

(a) Describe addition of chain maps and finite biproducts of complexes. Given a chain map $f : C \to D$, construct complexes K and Q with

$$K_n = \ker(f_n), \qquad Q_n = \operatorname{coker}(f_n),$$

and prove that they are respectively the kernel and cokernel of f in $\mathrm{Ch}(\mathcal{A})$.

(b) Prove that $\mathrm{Ch}(\mathcal{A})$ is abelian. Show that a sequence of complexes

$$0 \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$$

is short exact if and only if it is short exact in every degree.

(c) Put

$$Z_n(C) = \ker(d_n), \qquad B_n(C) = \operatorname{im}(d_{n+1}).$$

Construct the canonical monomorphism $B_n(C) \to Z_n(C)$, and define

$$H_n(C) = \operatorname{coker}\big(B_n(C) \longrightarrow Z_n(C)\big).$$

Show that chain maps induce morphisms on homology and that

$$H_n : \mathrm{Ch}(\mathcal{A}) \longrightarrow \mathcal{A}$$

is an additive functor.

Exercise 6. What chain homotopy remembers

Let C, D be chain complexes in an abelian category $\mathcal{A}$.

(a) Prove that chain homotopy is an equivalence relation on the set of chain maps $C \to D$. Show that it is compatible with addition and composition. Deduce that complexes and homotopy classes of chain maps form an additive category $K(\mathcal{A})$.

(b) Prove that homotopic chain maps induce the same morphisms on homology. Deduce that a chain homotopy equivalence is a *quasi-isomorphism*, meaning a chain map inducing isomorphisms on all homology objects.

(c) Work now in abelian groups. Let C be

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0,$$

concentrated in degrees $1, 0$, and let D have $\mathbb{Z}$ in degree 1 and zero elsewhere.

Describe all chain maps $C \to D$, and determine exactly when two are chain homotopic. Compute the abelian group

$$\mathrm{Hom}_{K(\mathbf{Ab})}(C, D).$$

Show that every chain map $C \to D$ induces zero on homology, although some are not nullhomotopic.

Exercise 7. Acyclic versus contractible

A complex C is *acyclic* if $H_n(C) = 0$ for every n, and *contractible* if 1_C is chain homotopic to zero.

(a) Prove that C is contractible if and only if it is acyclic and every short exact sequence

$$0 \longrightarrow Z_n(C) \longrightarrow C_n \xrightarrow{\bar{d}_n} Z_{n-1}(C) \longrightarrow 0$$

splits. Here $\bar{d}_n$ is the factorization of d_n through $Z_{n-1}(C)$; acyclicity makes it an epimorphism.

(b) Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\mathrm{mod}\,2} \mathbb{Z}/2 \longrightarrow 0,$$

with the displayed nonzero terms in degrees $2, 1, 0$. Show that this complex is acyclic but not contractible. Deduce that a quasi-isomorphism need not be a chain homotopy equivalence.

(c) Let C be a chain complex of vector spaces over a field. Show that C is chain homotopy equivalent to the complex having $H_n(C)$ in degree n and all differentials zero. Explain where choices enter your construction.

(d) Work with modules over the ring $\mathbb{Z}$. Consider the complex

$$C : \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0,$$

concentrated in degrees $1, 0$. Compute its homology, and let $H(C)$ denote its homology regarded as a complex with zero differentials.

Construct a quasi-isomorphism $C \to H(C)$, but prove that C and $H(C)$ are not chain homotopy equivalent. Thus the conclusion of part (c) can fail for modules over a ring that is not a field, even when every term of C is a free module.