Lecture 1: Additive and Abelian Categories

Algebraic Topology I

Conventions. Categories are locally small; $gf = g \circ f$. Rings are unital and modules are left modules. Biproduct inclusions and projections are denoted ι_i and ρ_i.

1 Categories and functors

Definition 1.1 (Category)**.** A *category* $\mathcal{C}$ consists of the following data:

1. A class $\mathrm{Ob}(\mathcal{C})$ of objects.
2. For every ordered pair of objects A, B, a set $\mathrm{Mor}_{\mathcal{C}}(A, B)$ of morphisms with source A and target B. A member is written $f\colon A \to B$. Source and target are part of the data of a morphism.
3. For every three objects A, B, C, a composition function
$$\mathrm{Mor}_{\mathcal{C}}(B, C) \times \mathrm{Mor}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(A, C), \qquad (g, f) \longmapsto g \circ f.$$
4. For every object A, a specified identity morphism $\mathrm{id}_A \in \mathrm{Mor}_{\mathcal{C}}(A, A)$.

The following diagrams are required to commute for every composable triple $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ (associativity) and every $f\colon A \to B$ (identity laws):

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g\circ f"'] & B \arrow[d, "h\circ g"] \\ C \arrow[r, "h"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[dr, "f"'] & A \arrow[d, "f"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f"'] & B \arrow[d, "\mathrm{id}_B"] \\ & B. \end{tikzcd}$$

A diagram commutes when the composites along paths with the same endpoints agree. A category is *small* if its objects also form a set. Examples are sets with functions and R-**Mod** with linear maps.

Definition 1.2 (Functor)**.** A functor $F\colon \mathcal{C} \to \mathcal{B}$ assigns an object $F(A)$ to each object A and a function $\mathrm{Mor}_{\mathcal{C}}(A, B) \to \mathrm{Mor}_{\mathcal{B}}(F(A), F(B))$, written $f \mapsto F(f)$, to every pair. It preserves identities and composition: the following diagrams commute.

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)"'] & F(B) \arrow[d, "F(g)"] \\ & F(C) \end{tikzcd} \qquad \begin{tikzcd} F(A) \arrow[r, bend left, "F(\mathrm{id}_A)"] \arrow[r, bend right, "\mathrm{id}_{F(A)}"'] & F(A). \end{tikzcd}$$

It is *faithful* if each of these functions is injective and *full* if each is surjective. A *full subcategory* is obtained by choosing some objects and retaining every morphism between them.

Definition 1.3 (Opposite category)**.** The opposite category $\mathcal{C}^{\mathrm{op}}$ has the same objects, with $\mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Mor}_{\mathcal{C}}(B, A)$ and composition reversed. A contravariant functor on $\mathcal{C}$ means a functor defined on $\mathcal{C}^{\mathrm{op}}$. For example, $\mathrm{Mor}_{\mathcal{C}}(A, -)$ is covariant, whereas $\mathrm{Mor}_{\mathcal{C}}(-, B)$ is contravariant.

Definition 1.4 (Natural transformation)**.** For functors $F, G\colon \mathcal{C} \to \mathcal{B}$, a natural transformation $\eta\colon F \Rightarrow G$ consists of morphisms $\eta_A\colon F(A) \to G(A)$ such that the following square commutes for every $f\colon A \to B$:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B). \end{tikzcd}$$

A natural isomorphism is a natural transformation whose components are isomorphisms.

Isomorphisms, monomorphisms, and epimorphisms

Definition 1.5. A morphism $f\colon A \to B$ is:

- an *isomorphism* if there is a morphism $g\colon B \to A$ with $gf = \mathrm{id}_A$ and $fg = \mathrm{id}_B$;
- a *monomorphism* (or *mono*) if, for every object X and every $u, v\colon X \to A$, the equality $fu = fv$ implies $u = v$;
- an *epimorphism* (or *epi*) if, for every object Y and every $u, v\colon B \to Y$, the equality $uf = vf$ implies $u = v$.

An inverse, when it exists, is unique: if g and h are inverses, then $g = g(fh) = (gf)h = h$.

Every isomorphism is both mono and epi, by cancellation with its inverse. The converse depends on the category.

Proposition 1.6 (Testing an isomorphism by maps into it)**.** *A morphism $f\colon A \to B$ is an isomorphism if and only if, for every object X, postcomposition induces a bijection*

$$f_*\colon \mathrm{Mor}_{\mathcal{C}}(X, A) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, B).$$

Proof. An inverse to f induces an inverse to each f_*. Conversely, surjectivity for $X = B$ gives $g\colon B \to A$ with $fg = \mathrm{id}_B$. Since $f(gf) = f = f\,\mathrm{id}_A$, injectivity for $X = A$ gives $gf = \mathrm{id}_A$. $\square$

2 Zero objects, products, and coproducts

Definition 2.1. An object I is *initial* if $\mathrm{Mor}_{\mathcal{C}}(I, X)$ is a singleton for every X. An object T is *terminal* if $\mathrm{Mor}_{\mathcal{C}}(X, T)$ is a singleton for every X. A *zero object* is both initial and terminal and is denoted 0.

Definition 2.2 (Zero morphism)**.** In a category with a zero object, the zero morphism $0_{A,B}\colon A \to B$ is the unique composite $A \to 0 \to B$. It does not depend on the choice of zero object. For $f\colon A \to B$, $g\colon X \to A$, and $h\colon B \to Y$,

$$f0_{X,A} = 0_{X,B}, \qquad 0_{A,B}g = 0_{X,B}, \qquad h0_{A,B} = 0_{A,Y}.$$

We usually write 0 when the source and target are clear.

Definition 2.3 (Product and coproduct)**.** A product of A_1, A_2 is an object P with projections $\rho_i\colon P \to A_i$ such that for every X and $f_i\colon X \to A_i$, there is a unique $u\colon X \to P$ making the left diagram commute. Dually, a coproduct is an object S with inclusions $\iota_i \colon A_i \to S$ such that for every Y and $g_i \colon A_i \to Y$, there is a unique $v \colon S \to Y$ making the right diagram commute:

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[d, dashed, "u"] \arrow[dr, "f_2"] & \\ A_1 & P \arrow[l, "\rho_1"] \arrow[r, "\rho_2"'] & A_2 \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r, "\iota_1"] \arrow[dr, "g_1"'] & S \arrow[d, dashed, "v"] & A_2 \arrow[l, "\iota_2"'] \arrow[dl, "g_2"] \\ & Y & \end{tikzcd}$$

Equivalently, the projections induce a bijection $\mathrm{Mor}_{\mathcal{C}}(X, P) \to \mathrm{Mor}_{\mathcal{C}}(X, A_1) \times \mathrm{Mor}_{\mathcal{C}}(X, A_2)$ for every X; the coproduct has the dual property.

Finite products include the empty product, a terminal object; finite coproducts include the empty coproduct, an initial object. Universal properties determine these objects up to unique isomorphism compatible with their structure maps.

3 Biproducts and the addition of morphisms

Definition 3.1 (Biproduct)**.** In a category with zero morphisms, a biproduct $A_1 \oplus A_2$ is an object equipped with inclusions $\iota_i \colon A_i \to A_1 \oplus A_2$ and projections $\rho_i \colon A_1 \oplus A_2 \to A_i$ such that the projections exhibit a product, the inclusions exhibit a coproduct, and

$$\rho_i \iota_j = \begin{cases} \mathrm{id}_{A_i} & i = j, \\ 0 & i \neq j. \end{cases}$$

A category is *semiadditive* if it has a zero object and all binary biproducts, hence all finite biproducts.

Theorem 3.2 (Canonical addition)**.** *In a semiadditive category, each* $\mathrm{Mor}_{\mathcal{C}}(A, B)$ *has a canonical commutative monoid structure. Its identity is the zero morphism, and composition preserves addition and zero in each variable.*

Proof. The diagonal $\Delta_A \colon A \to A \oplus A$ and codiagonal $\nabla_B \colon B \oplus B \to B$ are the unique maps making these diagrams commute for $i = 1, 2$:

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[dr, "\mathrm{id}_A"'] & A \oplus A \arrow[d, "\rho_i"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\iota_i"] \arrow[dr, "\mathrm{id}_B"'] & B \oplus B \arrow[d, "\nabla_B"] \\ & B. \end{tikzcd}$$

Given $f, g \colon A \to B$, let $s \colon A \oplus A \to B$ be the unique map whose restrictions to the two summands are f, g. Define

$$f + g = s\Delta_A = \nabla_B (f \oplus g)\Delta_A,$$

where $f \oplus g$ is uniquely specified by its diagonal components f, g and its two zero off-diagonal components.

If $g = 0$, then $s = f\rho_1$, so $f + 0 = f$; similarly $0 + f = f$. Interchanging the summands fixes Δ_A and exchanges the restrictions of s, proving commutativity.

Let $\Delta_A^{(3)} \colon A \to A \oplus A \oplus A$ have all three components id_A, and let $t \colon A \oplus A \oplus A \to B$ have restrictions f, g, h. Comparing components and restrictions shows that both $(f+g)+h$ and $f+(g+h)$ equal $t\Delta_A^{(3)}$.

For $k \colon B \to D$, the restrictions of ks are kf, kg, so $k(f + g) = kf + kg$. For $u \colon X \to A$, the restrictions of $s(u \oplus u)$ are fu, gu, and $\Delta_A u = (u \oplus u)\Delta_X$; hence $(f + g)u = fu + gu$. Composing a zero morphism with any morphism is zero. $\square$

Proposition 3.3 (Biproduct identities and matrices)**.** *In a semiadditive category,*

$$\iota_1\rho_1 + \iota_2\rho_2 = \mathrm{id}_{A_1\oplus A_2}\,.$$

More generally a morphism $f\colon \bigoplus_j A_j \to \bigoplus_i B_i$ *is uniquely determined by its entries* $f_{ij} = \rho_i f \iota_j$*, and*

$$f = \sum_{i,j} \iota_i f_{ij} \rho_j, \qquad (gf)_{kj} = \sum_i g_{ki} f_{ij}.$$

Proof. Composing the first asserted identity with either projection gives the corresponding projection, so it follows from uniqueness for products. Apply this identity to the source and target of f, then use bilinearity and $\rho_i\iota_j = \delta_{ij}$ to obtain both formulas. □

Additive categories and functors

Definition 3.4. A category is *preadditive* if every morphism set is equipped with an abelian group structure and composition is bilinear. It is *additive* if it is preadditive and has finite biproducts. Equivalently, it is semiadditive and all its canonical morphism monoids are groups.

The structures agree: the map s used above is $f\rho_1 + g\rho_2$, so $s\Delta_A = f + g$ also for the given addition.

Definition 3.5. A functor $F\colon \mathcal{A} \to \mathcal{B}$ between preadditive categories is *additive* if every induced map on morphism groups is a group homomorphism. In particular, $F(0) = 0$ on morphisms and $F(f+g) = F(f) + F(g)$.

Between additive categories, an additive functor preserves finite biproducts, because it preserves the biproduct identities. Conversely, a functor preserving finite products or finite coproducts preserves zero objects and the biproduct structure, and hence the formula defining addition in Theorem 3.2. Thus it is additive. The functors $\mathrm{Mor}_{\mathcal{A}}(M,-)\colon \mathcal{A} \to \mathbf{Ab}$ and $\mathrm{Mor}_{\mathcal{A}}(-,N)\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ are additive.

4 Kernels, cokernels, and abelian categories

Definition 4.1 (Kernel and cokernel)**.** In a category with zero morphisms, a kernel of $f\colon A \to B$ is a morphism $k\colon K \to A$ such that $fk = 0$ and, for every $u\colon X \to A$ with $fu = 0$, there is a unique $v\colon X \to K$ with $kv = u$.

A cokernel of f is a morphism $c\colon B \to Q$ such that $cf = 0$ and, for every $u\colon B \to Y$ with $uf = 0$, there is a unique $v\colon Q \to Y$ with $vc = u$.

We also write $\ker f = K$ and $\operatorname{coker} f = Q$ for the corresponding objects. For modules, $K = f^{-1}(0)$ and $Q = B/f(A)$.

Lemma 4.2. *Kernels are monomorphisms and cokernels are epimorphisms. In an additive category,* f *is mono if and only if* $fu = 0$ *implies* $u = 0$ *for all* u *with target* A*. Dually,* f *is epi if and only if* $uf = 0$ *implies* $u = 0$ *for all* u *with source* B*. If a kernel of* f *exists,* f *is mono precisely when its kernel object is zero; dually for epis and cokernels.*

Proof. If $kv = kw$ for a kernel k of f, both v, w are factorizations of the same morphism through k, so uniqueness gives $v = w$. The cokernel assertion is dual. In an additive category, $fu = fv$ is equivalent to $f(u - v) = 0$. This proves the cancellation criterion. If the kernel object is zero, every morphism killed by f is zero. Conversely, if f is mono, $fk = 0$ forces $k = 0$; since k is mono, $k\,\mathrm{id}_K = k0$ forces $\mathrm{id}_K = 0$, which makes K a zero object. □

Definition 4.3 (Abelian category)**.** A category $\mathcal{A}$ is *abelian* if:

1. $\mathcal{A}$ is additive;
2. every morphism has a kernel and a cokernel;
3. every monomorphism is a kernel of some morphism, and every epimorphism is a cokernel of some morphism.

The definition is self-dual. Examples are **Ab**, R-**Mod**, and finite-dimensional vector spaces.

5 The canonical isomorphism from coimage to image

Lemma 5.1. *In an abelian category, every mono is a kernel of its own cokernel, and every epi is a cokernel of its own kernel.*

Proof. Let $m\colon M \to B$ be mono, so m is a kernel of some $g\colon B \to D$. Let $c\colon B \to Q$ be a cokernel of m. Since $gm = 0$, write $g = tc$. If $cu = 0$, then $gu = tcu = 0$, so u factors uniquely through m. This proves that m is a kernel of c. Reverse arrows for the epi statement. ☐

Proposition 5.2. *A morphism in an abelian category that is both mono and epi is an isomorphism.*

Proof. If $f\colon A \to B$ is mono, it is a kernel of some $g\colon B \to D$. Since $gf = 0$ and f is epi, $g = 0$. The kernel property applied to id_B gives $s\colon B \to A$ with $fs = \mathrm{id}_B$. Monicity and $f(sf) = f$ then give $sf = \mathrm{id}_A$. ☐

Lemma 5.3 (Pullbacks of epimorphisms)**.** *If $e\colon E \to B$ is epi and $t\colon X \to B$ is any morphism in an abelian category, their pullback exists and its projection $p\colon P \to X$ is epi.*

Proof. Put $r = e\rho_1 - t\rho_2\colon E \oplus X \to B$, and let $s\colon P \to E \oplus X$ be its kernel. Set $a = \rho_1 s$ and $p = \rho_2 s$. Then $ea = tp$. For maps $u\colon Y \to E$, $v\colon Y \to X$, let $b\colon Y \to E \oplus X$ be the unique map with components u, v. The equality $eu = tv$ is equivalent to $rb = 0$, so the kernel property gives a unique $Y \to P$ inducing u, v. This is the pullback universal property.

The map r is epi since $r\iota_1 = e$. By Lemma 5.1, it is a cokernel of s. If $w\colon X \to Y$ satisfies $wp = 0$, then $w\rho_2 s = 0$, so there is $z\colon B \to Y$ with $w\rho_2 = zr$. Restricting to E gives $ze = 0$, whence $z = 0$. Restricting to X now gives $w = 0$. The epi criterion in Lemma 4.2 proves that p is epi. ☐

Definition 5.4 (Image, coimage, and comparison)**.** For $f\colon A \to B$, choose its kernel $k\colon K \to A$ and cokernel $c\colon B \to Q$. Define the coimage and image together with their structure maps by

$$q\colon A \longrightarrow \operatorname{coim} f = \operatorname{coker} k, \qquad m\colon \operatorname{im} f = \ker c \longrightarrow B.$$

There is a unique morphism $\overline{f}\colon \operatorname{coim} f \to \operatorname{im} f$ such that $f = m\overline{f}q$:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "q"'] & B \\
\operatorname{coim} f \arrow[r, "\overline{f}"'] & \operatorname{im} f. \arrow[u, tail, "m"']
\end{tikzcd}$$

Indeed, $cf = 0$ first gives a unique $u\colon A \to \operatorname{im} f$ with $mu = f$. Then $muk = fk = 0$ and monicity of m give $uk = 0$, so u factors uniquely through q.

Theorem 5.5 (First isomorphism theorem)**.** *In an abelian category the canonical morphism $\overline{f}\colon \operatorname{coim} f \to \operatorname{im} f$ is an isomorphism. In particular, every morphism factors as an epimorphism followed by a monomorphism.*

Proof. We first prove that $\overline{f}$ is mono. Let $j\colon L \to \operatorname{coim} f$ be its kernel, and form the pullback

$$\begin{tikzcd}
P \arrow[r, "a"] \arrow[d, two heads, "p"'] & A \arrow[d, two heads, "q"] \\
L \arrow[r, tail, "j"'] & \operatorname{coim} f.
\end{tikzcd}$$

The projection p is epi by Lemma 5.3. Since $qa = jp$ and $\overline{f}j = 0$,

$$fa = m\overline{f}qa = m\overline{f}jp = 0.$$

By the kernel property of k, write $a = kv$. Then $jp = qa = qkv = 0$. Since p is epi, $j = 0$. Because j is mono, its domain L is a zero object. Thus $\overline{f}$ has zero kernel and is mono.

Apply this same argument in $\mathcal{A}^{\mathrm{op}}$. The coimage of the reversed morphism f^{op} is the image of f, its image is the coimage of f, and its comparison is the reversed arrow $\overline{f}$. The argument proves this reversed arrow mono, which says precisely that $\overline{f}$ is epi in $\mathcal{A}$. Proposition 5.2 now makes $\overline{f}$ an isomorphism. Hence $f = m(\overline{f}q)$ is an epi followed by a mono. $\square$

For modules, $\operatorname{coim} f = A/\ker f$, $\operatorname{im} f = f(A) \subseteq B$, and $\overline{f}(a + \ker f) = f(a)$.

6 Exact sequences and chain complexes

Definition 6.1 (Exactness)**.** For $A \xrightarrow{f} B \xrightarrow{g} C$ with $gf = 0$, there is a canonical morphism $\operatorname{im} f \to \ker g$ compatible with their inclusions into B. The pair is *exact at B* if this map is an isomorphism. A sequence is exact if it is exact at each object where two adjacent maps are specified. A short exact sequence is an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Equivalently, f is a kernel of g and g is a cokernel of f.

An additive functor is *left exact* if it preserves kernels, *right exact* if it preserves cokernels, and *exact* if it preserves short exact sequences.

Definition 6.2 (Chain complex and chain map)**.** A chain complex $C_\bullet$ in an abelian category consists of objects C_n, $n \in \mathbb{Z}$, and differentials $d_n^C\colon C_n \to C_{n-1}$ satisfying $d_{n-1}^C d_n^C = 0$. A chain map $f\colon C_\bullet \to D_\bullet$ consists of maps $f_n\colon C_n \to D_n$ making the following squares commute:

$$\begin{tikzcd}
C_n \arrow[r, "d_n^C"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_n \arrow[r, "d_n^D"'] & D_{n-1}
\end{tikzcd} \qquad (n \in \mathbb{Z}).$$

Composition and identities are defined degreewise; the resulting category is denoted $\mathbf{Ch}(\mathcal{A})$.

The cycle object is $Z_n(C) = \ker d_n^C$, and the boundary object is $B_n(C) = \operatorname{im} d_{n+1}^C$. The equation $d_n^C d_{n+1}^C = 0$ gives a canonical mono $B_n(C) \to Z_n(C)$. Define

$$H_n(C) = \operatorname{coker}\big(B_n(C) \longrightarrow Z_n(C)\big).$$

A chain map induces maps on cycles and boundaries by their universal properties and therefore induces $H_n(f)\colon H_n(C) \to H_n(D)$. The category $\mathbf{Ch}(\mathcal{A})$ is abelian, with kernels and cokernels computed degreewise; the induced differentials exist uniquely by the corresponding universal properties. A complex is *acyclic* if every $H_n(C)$ is zero, equivalently if its underlying sequence is exact in every degree.

Definition 6.3 (Chain homotopy)**.** For chain maps $f, g\colon C_\bullet \to D_\bullet$, a chain homotopy from g to f is a family $h_n\colon C_n \to D_{n+1}$ such that

$$f_n - g_n = d_{n+1}^D h_n + h_{n-1} d_n^C \qquad (n \in \mathbb{Z}).$$

A complex is *contractible* if its identity chain map is chain homotopic to zero.

Proposition 6.4. *Chain-homotopic maps induce the same maps on homology. In particular every contractible complex is acyclic.*

Proof. Let $z\colon Z_n(C) \to C_n$ be the kernel inclusion. Since $d_n^C z = 0$, the homotopy identity restricts to $(f_n - g_n)z = d_{n+1}^D h_n z$. This factors through $B_n(D)$ and becomes zero in $H_n(D)$. Thus $H_n(f) = H_n(g)$. If $f = \mathrm{id}_C$ and $g = 0$, the identity of $H_n(C)$ is zero, so $H_n(C)$ is a zero object. $\square$

The converse fails for modules over a general ring. The exact complex of abelian groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

(in degrees $2, 1, 0$) is not contractible: a contracting homotopy in degree 0 would give a section $\mathbb{Z}/2 \to \mathbb{Z}$ of the quotient map, which is impossible. Over a field, every acyclic complex is contractible after choosing splittings of its short exact sequences of cycles.