Cohomology, Spectra, and Filtrations

Algebraic Topology II | Proposed course | October 22, 2026

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1 Suspension and Represented Cohomology

A cohomology theory assigns invariants in all integer degrees, related by suspension. Spectra organize those degree shifts into one object. We begin with the conventions needed for that construction; representability and the category of spectra will be developed in the later notes. The second section gives a complete algebraic example of the filtration methods used to compute such invariants.

All spaces in this section are based compactly generated spaces, and mapping and loop spaces use the compactly generated topology. Write $[X, Y]_*$ for based homotopy classes. The reduced suspension is $\Sigma X = S^1 \wedge X$, and ΩY is the based loop space. The suspension–loop adjunction gives a natural bijection

$$[\Sigma X, Y]_* \cong [X, \Omega Y]_*.$$

Indeed, a based map out of $S^1 \wedge X$ is the same as a based map from X into the based mapping space from S^1. This correspondence also identifies based homotopies, so it descends to the displayed homotopy classes.

1.1 A Sequential Description

For an initial description, a prespectrum consists of based spaces E_j for $j \geq 0$ and based structure maps

$$\sigma_j : \Sigma E_j \longrightarrow E_{j+1}.$$

Its adjoint is a map $\epsilon_j : E_j \to \Omega E_{j+1}$. Call this description a *homotopy Ω-prespectrum* when each ϵ_j is a based homotopy equivalence. This is a deliberately stronger hypothesis than a weak equivalence of spaces; passage to a standard homotopy-theoretic model of spectra will be treated separately.

For an integer q and a based space X, define the candidate reduced cohomology set by

$$\widetilde{E}^q(X) = \operatorname*{colim}_{k \geq \max(0, -q)} [\Sigma^k X, E_{k+q}]_*.$$

The transition map sends the class of f to the class of $\sigma_{k+q} \circ \Sigma f$. This defines a contravariant functor in X because precomposition commutes with suspension and with the structure maps. For a homotopy Ω-prespectrum every transition map is a bijection: under adjunction it is the map induced by the homotopy equivalence ϵ_{k+q}.

Taking the cofinal tail $k \geq 2$ supplies abelian group structures. Maps out of a double suspension form an abelian group, by the two compatible concatenations and the Eckmann–Hilton argument. The transition maps preserve these structures. Consequently $\widetilde{E}^q(X)$ is an abelian group, naturally in X.

The shift convention can be checked without a representability theorem. Reindexing by $l = k + 1$ gives

$$\widetilde{E}^q(\Sigma X) \cong \widetilde{E}^{q-1}(X),$$

since both expressions are colimits of $[\Sigma^l X, E_{l+q-1}]_*$ over cofinal tails. Thus positive cohomological degree corresponds to positive indexing of the representing spaces, whereas suspension lowers the cohomological index in this formula.

1.2 Exactness and the Remaining Construction

Let (X, A) be a based CW pair, with its basepoint in A, and let X/A be its based cofiber. The subcomplex inclusion is a cofibration, so this quotient models its based homotopy cofiber. The Puppe sequence supplies exact sequences of based mapping sets into any based space. Here exactness means that a class maps to the constant class if and only if it comes from the preceding term. After sufficient suspension these are exact sequences of abelian groups. Taking the filtered colimit, which is exact for abelian groups, gives the cofiber exact sequence for the groups just defined. In particular, the segment at $\widetilde{E}^q(X)$ is

$$\widetilde{E}^q(X/A) \longrightarrow \widetilde{E}^q(X) \longrightarrow \widetilde{E}^q(A).$$

The connecting map has target $\widetilde{E}^{q+1}(X/A)$, as the suspension convention above requires.

This argument explains suspension and exactness. It does not by itself prove a general representability theorem or construct smash products of spectra. Those require the subsequent development. Likewise, an arbitrary prespectrum should not be used to infer a wedge axiom by commuting a filtered colimit with an infinite product. The homotopy Ω-prespectrum hypothesis identifies the colimit with any sufficiently high stage, where the based wedge mapping property can instead be applied directly.

2 The Spectral Sequence of a Finite Filtration

A filtration separates a chain complex into simpler quotients.[1] Their homology gives a first approximation to the homology of the whole complex. A class surviving in one quotient may still have a boundary in a lower filtration level; the successive differentials record that boundary.

Let $C_\bullet$ be a chain complex of modules over a ring R, with differential $\partial : C_n \to C_{n-1}$. Assume it has an increasing filtration by subcomplexes

$$0 = F_{-1}C_\bullet \subseteq F_0 C_\bullet \subseteq \cdots \subseteq F_N C_\bullet = C_\bullet.$$

Set $F_p C = 0$ for $p < 0$ and $F_p C = C$ for $p \geq N$. No splitting of this filtration is assumed. The integer p measures filtration; $p + q$ is homological degree. A finite filtration need not give a first-quadrant spectral sequence.

[1] For the standard filtered-complex and exact-couple constructions, compare Haynes Miller, *Algebraic Topology II*, MIT 18.906, Lectures 23 and 25, in the official course notes.

2.1 The Exact Couple

The short exact sequences of complexes

$$0 \longrightarrow F_{p-1}C \longrightarrow F_pC \longrightarrow F_pC/F_{p-1}C \longrightarrow 0$$

give long exact homology sequences. Define

$$D^1_{p,q} = \mathrm{H}_{p+q}(F_pC), \qquad E^1_{p,q} = \mathrm{H}_{p+q}(F_pC/F_{p-1}C).$$

The three types of maps in those exact sequences have degrees

$$\begin{aligned} i &: D^1_{p,q} \longrightarrow D^1_{p+1,q-1}, \\ j &: D^1_{p,q} \longrightarrow E^1_{p,q}, \\ k &: E^1_{p,q} \longrightarrow D^1_{p-1,q}. \end{aligned}$$

Thus i is inclusion on homology, j is quotient, and k is the connecting homomorphism. These groups and maps form an exact couple: the image of each map is the kernel of the next, with the displayed degree shifts understood.

The first differential is $d^1 = jk$, so

$$d^1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}.$$

It squares to zero because $kj = 0$. More concretely, lift a quotient cycle to $x \in F_pC_{p+q}$. Its boundary lies in $F_{p-1}C$, and $d^1[x]$ is represented by ∂x modulo $F_{p-2}C$. This also fixes the connecting-map sign convention for these notes.

2.2 Deriving the Couple

Suppress the degree indices temporarily. For an exact couple (D, E, i, j, k), set $D' = \operatorname{im} i$ and $E' = \ker(jk)/\operatorname{im}(jk)$. Define

$$i'(ix) = i^2x, \qquad j'(ix) = [jx], \qquad k'([e]) = ke.$$

These formulas are well-defined. If $ix = iy$, then $x - y \in \ker i = \operatorname{im} k$, so $jx - jy$ is a jk-boundary. If $jk(e) = 0$, then $ke \in \ker j = \operatorname{im} i$. Changing e by $jk(f)$ does not change ke, since $kj = 0$.

For completeness, exactness can be checked directly. If $j'(ix) = 0$, write $jx = jk(e)$; then $x - ke \in \ker j = \operatorname{im} i$, and ix lies in $\operatorname{im} i'$. If $k'([e]) = 0$, write $e = jx$; then $[e] = j'(ix)$. Finally, if $i'(ix) = 0$, write $ix = ke$; the equality $jk(e) = j(ix) = 0$ makes $[e]$ a class in E', and $ix = k'([e])$. The reverse inclusions follow from the zero composites.

Iterating this construction gives $E^{r+1} = \mathrm{H}(E^r, d^r)$. Restoring the indices gives

$$d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

The total degree decreases by one. The filtration drop is r, because surviving the preceding differentials permits the boundary of a representative to be moved into $F_{p-r}C$.

2.3 Representatives and the Target

The following formulas make the last assertion and the finite-filtration target explicit. For $r \geq 0$, let

$$Z^r_{p,n} = \{x \in F_pC_n : \partial x \in F_{p-r}C_{n-1}\}.$$

Since the filtration is by subcomplexes, $Z^0_{p,n} = F_pC_n$. For $r \geq 1$, the page obtained from the exact couple has the representative description

$$E^r_{p,q} = \frac{Z^r_{p,p+q}}{Z^{r-1}_{p-1,p+q} + \partial Z^{r-1}_{p+r-1,p+q+1}}.$$

The first page is the homology of the quotient complex. The formula for d^r sends the class of x to the class of ∂x. If this class is zero, write $\partial x = z + \partial y$ with $z \in Z^{r-1}_{p-r-1,p+q-1}$ and $y \in Z^{r-1}_{p-1,p+q}$. Replacing x by $x - y$ puts its boundary in $F_{p-r-1}C$. Thus cycles on page r have representatives in $Z^{r+1}_{p,p+q}$. For two representatives in $Z^{r+1}_{p,p+q}$ whose difference is $z + \partial y$ in the old denominator, taking boundaries shows that z lies in $Z^r_{p-1,p+q}$. The old boundaries and incoming d^r boundaries together are $\partial Z^r_{p+r,p+q+1}$. This proves the displayed formula for the next page.

For $0 \leq p \leq N$ and $r \geq N + 1$, that quotient becomes

$$\frac{F_pC_n \cap \ker \partial}{(F_{p-1}C_n \cap \ker \partial) + (F_pC_n \cap \operatorname{im} \partial)}.$$

Define $F_p\mathrm{H}_n(C)$ to be the image of $\mathrm{H}_n(F_pC) \to \mathrm{H}_n(C)$. Sending a cycle to its homology class identifies the last quotient with

$$E^\infty_{p,q} \cong F_p\mathrm{H}_{p+q}(C)/F_{p-1}\mathrm{H}_{p+q}(C).$$

Indeed, a cycle in F_pC_n maps into $F_{p-1}\mathrm{H}_n(C)$ exactly when it differs from a cycle in $F_{p-1}C_n$ by a boundary in C_n. This proves the claimed kernel and surjectivity. The filtration is finite and exhaustive, so no infinite-filtration convergence assertion is needed. The associated graded groups do not determine the extensions between filtration levels.

3 A Differential on the Second Page

Fix an integer $m \geq 2$. Let $C_2 = \mathbb{Z}\{a\}$, $C_1 = \mathbb{Z}\{b\}$, and $C_n = 0$ in all other degrees, with $\partial a = mb$. Give it the filtration

$$F_0C = F_1C = \mathbb{Z}\{b\}[1], \qquad F_2C = C.$$

Here $\mathbb{Z}\{b\}[1]$ means the complex generated by b in degree one with zero differential. This is an algebraic filtration, not the skeletal filtration of a CW complex.

The only nonzero first-page groups are $E^1_{2,0} = \mathbb{Z}\{[a]\}$ and $E^1_{0,1} = \mathbb{Z}\{[b]\}$. The first differential is zero: the boundary mb lies two filtration levels below a. Consequently the same two groups occur on page two. The representative formula gives

$$E^2_{2,0} = \mathbb{Z}\{[a]\} \xrightarrow{d^2 = m} E^2_{0,1} = \mathbb{Z}\{[b]\}.$$

Thus the third page has the single nonzero group $E^3_{0,1} = \mathbb{Z}/m\mathbb{Z}$, and all later differentials vanish. Directly, multiplication by m is injective on $\mathbb{Z}$, so $\mathrm{H}_2(C) = 0$ and $\mathrm{H}_1(C) = \mathbb{Z}/m\mathbb{Z}$. The spectral sequence has recovered these groups, with no extension ambiguity because only one filtration quotient is nonzero.

One can also see this differential in the derived couple. The connecting map sends $[a]$ to $m[b] \in \mathrm{H}_1(F_1C)$. Its quotient in F_1C/F_0C is zero, giving $d^1[a] = 0$. Since $F_0C \to F_1C$ is the identity on the copy of $\mathbb{Z}\{b\}$, lift the connecting class to $m[b] \in \mathrm{H}_1(F_0C)$. Projecting to the quotient $F_0C/F_{-1}C$ gives $d^2[a] = m[b]$.