Generalized Cohomology, Spectra, and Spectral Sequences

Algebraic Topology II | Proposed course | October 22, 2026

Contents

1 The Second Course **1**
1.1 Prerequisites and the First New Proofs 1
1.2 Conventions to Fix at the Start 1

2 A Dependency-Ordered Lecture Sequence **2**

3 Placement of the Existing Appendices **3**
3.1 First-Course Main Sections 3
3.2 Mapping Complexes and Künneth 3

4 Cross-References and Preservation **4**
4.1 References That Need Explicit Attention 4
4.2 A Safe Editing Order 4

1 The Second Course

The first course constructs ordinary homology and develops elementary homotopy theory. The second course should explain how cohomology theories arise from spectra and how filtrations make their values computable. Ordinary cohomology, stable homotopy, and complex topological K-theory provide recurring examples.

The outline below is a proposal, not completed lecture notes. It uses the current twelve-lecture Math 615 manuscript, including its five appendices. No current appendix has been removed or rewritten. The elementary degree material remains in Lecture 8, before cellular calculations that use it.

1.1 Prerequisites and the First New Proofs

Students need chain complexes, tensor products, Tor and Ext, singular homology, cellular homology, homotopy pushouts, homotopy groups, fibrations, and covering spaces. These are already developed in Math 615. Ordinary cohomology and its products should be constructed in the second course, rather than silently assumed.

Lecture 9 states Whitehead and Hurewicz without proof and expressly defers their proofs to Math 616. The second course should therefore supply cellular approximation, relative homotopy groups, the relative connectivity arguments, and proofs of those two theorems before relying on them in stabilization. This is a genuine addition, not material already contained in an appendix.

1.2 Conventions to Fix at the Start

Use based CW complexes for the initial homotopy constructions and specify the category of spaces before introducing mapping spaces. Use homological grading for chain complexes and cohomological grading for ordinary and generalized cohomology. Define the suspension

convention once and check every spectral-sequence differential against it. Write $\boxtimes$ for the external homology product and $\times$ for products of spaces. Preserve the existing symbols for cones, cylinders, and morphism objects. Label a comparison map as an equivalence only after proving that it is one.

2 A Dependency-Ordered Lecture Sequence

The numbered units below are substantial lecture-note sections, not a promise of one classroom meeting each. Units 1–12 form the core. The review material in the next section can be assigned as reading without displacing the central development.

1. **CW approximation and the comparison theorems.** Relative homotopy, cellular approximation, Whitehead, and Hurewicz, with the needed connectivity arguments proved. Revisit sphere degree only by reference to Math 615, §8.1.3.
2. **Ordinary cohomology and products.** Cochains, the universal coefficient theorem, relative cohomology, cup products, and the relation between cup and external products. Work through circles, surfaces, and projective spaces.
3. **Derived tensor products and mapping objects.** Mapping complexes, chain homotopies, cones, and derived functors. Import Appendix A with its signs and proofs. Derive the ordinary Künneth theorem and explain both its natural exact sequence and its noncanonical splitting over a principal ideal domain.
4. **Generalized cohomology and representability.** Reduced theories, suspension, exactness, wedge conditions, Eilenberg–Mac Lane spaces, and representability on based CW complexes. State the precise representability hypotheses before applying them. Separate representability by spaces from the compatible deloopings defining a spectrum.
5. **Spectra and stable homotopy.** Suspension spectra, infinite loop spaces, stable homotopy groups, and the stabilization map. Prove the suspension–loop adjunction in the chosen framework. Include the relevant Freudenthal range, rather than treating unstable groups as already stable.
6. **Cofiber sequences and represented theories.** Fibers and cofibers of spectra, suspension and desuspension, long exact sequences, and the formulas for the homology and cohomology represented by a spectrum. Compare ordinary homology with the Eilenberg–Mac Lane spectrum. Revisit the chain-cone model from Unit 3 as an example, not a definition of all spectra.
7. **Products, modules, and operations.** Smash products and function spectra; ring and module spectra; cup products and natural transformations between represented theories. Use complex K-theory as a principal example, identifying explicitly where Bott periodicity is supplied as a theorem. Do not silently claim a proof of Bott periodicity in an elementary construction of K-theory.
8. **Filtered Objects and Spectral Sequences.** Construct the spectral sequence of a filtered complex and then of a filtered spectrum. Prove the differential bidegrees, identify the associated graded of the target, and distinguish convergence from the extension problem. Begin with finite filtrations; treat infinite filtrations with stated convergence hypotheses.
9. **The Atiyah–Hirzebruch Spectral Sequence.** Filter a CW complex by skeleta. Derive the homological and cohomological sequences, their coefficient systems in this skeletal construction, and multiplicative structure when the representing spectrum has a product. Compute examples with sphere, ordinary-cohomology, and K-theory coefficients, recording unresolved differentials and extensions rather than guessing them.
10. **Fibrations and the Serre Spectral Sequence.** Construct the homology and cohomology

sequences with their local coefficient systems. Explain the role of the fundamental-group action. Compute the Hopf fibration and sphere-bundle examples. Compare this filtration with the skeletal construction in Unit 9; these are different constructions with related examples.

11. **Künneth and Change of Coefficients.** Explain the relation between the ordinary Künneth short exact sequence and spectral sequences for derived tensor products. Introduce the module-spectrum analogue with explicit connectivity or convergence hypotheses. Return to the torsion examples from Appendix C.
12. **Orientations and Thom Isomorphisms.** Thom spaces, ordinary and generalized orientations, Thom classes, and the Thom isomorphism. Use the coefficient theory to explain which orientations exist. Relate the computations to the preceding spectral sequences.

A further unit on the Adams spectral sequence is a natural continuation, not part of the minimum core. It requires additional work on resolutions, cohomology operations, completion, and convergence. It should not be inserted as an unexplained final example.

3 Placement of the Existing Appendices

The appendices do not all have the same role. Appendices A and C directly support stable homotopy and generalized cohomology. Appendices B, D, and E are geometric applications of first-course homology. The latter three should become main first-course sections, not be presented as consequences of spectra.

3.1 First-Course Main Sections

Appendices B, D, and E should become substantive first-course sections, rather than being forced into the spectra curriculum. Appendix B belongs after the cellular computations in Lecture 8. Appendix D belongs with the geometric applications of homology and degree. Appendix E should follow Appendix D's material, with its complete odd-degree and hemisphere-chain argument retained. These placements are proposals for a later reorganization; the current notes remain untouched.

In detail, preserve Appendix B, §§B.1–B.4: attaching maps, surfaces, Euler characteristic, and examples. Preserve Appendix D, §§D.1–D.2: the no-retraction obstruction, ray proof of Brouwer's theorem, local homology, invariance of dimension, and manifold dimension. Preserve Appendix E, §§E.1–E.3: reduction to boundary degree, the odd-map lemma with its chain proof, and consequences for embeddings and manifolds. Each remains available to the second course by a reference to the first-course section.

3.2 Mapping Complexes and Künneth

Transfer Appendix A, §A.1, into Unit 3 before derived mapping objects: the tensor–Mor adjunction, interval complex, chain-homotopy sign, and zeroth-homology description of homotopy classes. Transfer §A.2 immediately afterward: the homotopy square, comparison map, quasi-isomorphism criterion, and the signed Mayer–Vietoris sequence. Preserve all seven existing labelled statements and their proofs.

Transfer Appendix C, §§C.1–C.2, into Unit 3 after the tensor-product construction and its relation to singular chains has been established. Keep the field theorem, the principal-ideal-domain exact sequence, the degree of the Tor term, and the explanation of splitting. Transfer §§C.3–C.4 into the examples following that theorem: circle factors, products of surfaces, the product of two real projective planes, product maps on top homology, and factor-interchange

signs. Unit 11 refers back to this ordinary theorem before giving the module-spectrum analogue. Do not state the ordinary principal-ideal-domain splitting as a general fact about spectra.

4 Cross-References and Preservation

The current unified course remains the published source until the replacement chapters exist, are checked, and are linked. Distribution between volumes must not make a cited proof disappear from a student's accessible notes.

4.1 References That Need Explicit Attention

Math 615 Lecture 4 cites Appendix B, Proposition B.1.2, and the projective-plane calculation. Replace that reference with a stable labelled destination in the first-course main section, or keep the short attaching-word proof in Lecture 8 and refer there. Retain the existing comparison to Lecture 8, Theorems 8.2.2 and 8.3.4.

The Lecture 8 PDF metadata still mentions Appendix A. The extraction scripts also attach Appendix A to the Lecture 8 and Lectures 7–8 reading copies. When the relocation is approved, replace that supplemental attachment with the corresponding exact extract of Unit 3; do not remove the supporting material without a replacement. Preserve the separately included Lecture 6 prism-construction pages, which are not one of Appendices A–E.

Appendix A refers back to Lecture 2, Lecture 3, and Lecture 7. Those become explicit references to *Algebraic Topology I*, with volume and statement labels, rather than unqualified lecture numbers. The same treatment applies to degree and sphere-homology references in the geometric applications sections. The second course's proofs of Whitehead and Hurewicz should be linked from the existing statements in Math 615, §9.4, without changing the status of their earlier uses.

4.2 A Safe Editing Order

First freeze the current PDF and editable source archive. Create the second-course sources separately and retain the existing theorem labels in a relocation index. Copy each appendix block, including its proof and examples, into the destination given above. Compile the new volume and its extracts; check references, displayed signs, coefficients, and visible formatting. Only then replace the first volume's appendix entries with links or approved short prerequisite extracts. Keep the full original version available.

The relocation index should record the old appendix section, every statement label, its new chapter and section, and the old and new PDF destinations. A content comparison must account for prose computations as well as numbered statements. The current degree subsection and the polished subdivision and small-chain arguments are outside this relocation and must remain untouched.